Graph Curves

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Abstract

Graph curves are a useful construction for considering various problems in algebraic geometry. In this paper, we explain some of the results about graph curves from the seminal paper by Bayer and Eisenbud [3]. We provide background in simplicial cohomology and the canonical series of a curve for the beginning reader.

In our study of algebraic curves, it is helpful to consider degenerations – objects that are simpler to characterize, while still having a lot of explanatory power. In this paper, we will investigate graph curves, in a sense the simplest degeneration one could use.

Definition 0.1. A graph curve is a connected union of projective lines, each line meeting three others in ordinary double points. The lines and their intersections can be recorded in a dual graph $G = (V, E)$, where each vertex of V corresponds to a line, and each edge e connecting v_1 and v_2 represents a node where the corresponding lines intersect. $C(G)$ is used to denote the curve corresponding to a graph G.

There are many ways to relax the definition of a graph curve, allowing some more general statements while giving up some of the simplicity and symmetry of the geometry. For instance, instead of each line intersecting three others, lines may be allowed to intersect one or two lines instead, as in [4]. Additionally, instead of allowing only lines in the curve, we may allow rational curves corresponding to the vertices, as in [1] and [2].

Example 0.2 (Bayer-Eisenbud, Introduction)**.** Consider a set of four distinct lines in the plane. For simplicity take the zero sets of the polynomials X, Y, Z, and $(X + Y + Z)$, which (like all other lines in $\mathbb{P}^2$) intersect. The corresponding graph is K_4, the complete graph on four vertices, since we have four lines that intersect each of the others. This graph is trivalent.

The polynomial defining this curve is trivially $P(X, Y, Z) = XYZ(X + Y + Z)$.

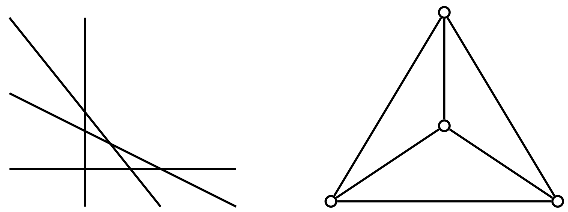

Figure 1: An arrangement of lines, and the corresponding dual graph.

This relatively trivial calculation leads us to the central agenda of the paper:

Question 0.3. What are the generators of the canonical ideal of an arbitrary graph curve?

Graph curves have the property of being *stable*; the presence of three nodes on each component line causes the isomorphism group of the curve to be finite.

The fact that makes them particularly useful in the study of curves is that they are situated on the boundary of the moduli space of curves. When a property can be proven to hold for a graph curve, this can sometimes be extended to all the other curves of the same genus.

Bayer and Eisenbud used this to investigate Green's Conjecture about the Clifford Index of a curve [3]. Ciliberto, Harris, and Miranda also took this approach to prove the surjectivity of the Wahl map [5]. This exposition will mainly follow the first three sections of the paper of Bayer and Eisenbud, culminating in the computation of the canonical ideal of a graph curve.

Acknowledgements: I would like to thank Jessica Sidman for initially introducing me to the topic, and answering questions over the course of preparation of this paper. I also thank Bernd Sturmfels for his advice in pursuing this research and for referring me to the pre-print by Lohne ([9]).

1 Basic Properties

Before we approach the canonical ideal, let us state some properties that arise from the basic definitions.

Proposition 1.1 (Properties of the Graph)**.** *The graph will have the following properties:*

1. *Trivalent*

2. *Undirected (though we will impose an orientation later)*

3. *Simple*

4. *Connected*

5. $|V| = 2n$*, and* $|E| = 3n$ *(for some integer* $n \geq 2$*).*

Proof. 1. Three intersections per line correspond to three edges incident to each vertex.

2. The intersection data of the lines does not dictate any orientation

3. If two vertices would be connected to two edges, the corresponding lines would have intersection $\ell_1 \cap \ell_2$ with at least two points, which implies $\ell_1 = \ell_2$. Therefore, only one edge can be allowed.

4. We required that the curve be connected, so the graph must be as well.

5. In any graph, $\sum \deg v_i = 2|E|$ (Handshaking Lemma). For a trivalent graph, that means $3|V| = 2|E|$.

$\square$

Remark 1.2. Not every trivalent graph corresponds to a graph curve; e.g. consider the graph in Figure 2. In the subgraph imposed on vertices $v_1, \ldots, v_4$, the lines $C(v_2)$ and $C(v_4)$ span a $\mathbb{P}^2$. $C(v_1)$ and $C(v_3)$ must be contained in the same plane, since they intersect both lines. Bezout's Theorem dictates that they intersect; however, the graph indicates otherwise.

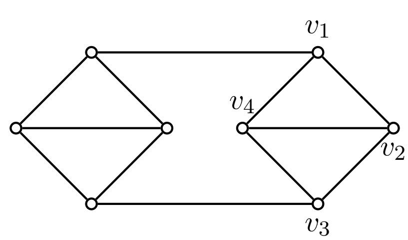

Figure 2: Trivalent graph without corresponding graph curve

Conjecture 1.3. Given any connected, trivalent, simple graph G, the graph can be realized as a graph curve $C(G)$, as long as G does not have $K_4 \setminus \{e\}$ as an induced subgraph.

The intuition for the above is that when two lines are not forced into the same $\mathbb{P}^2$ there's enough space to allow for intersection or lack of intersection, as desired.

Proposition 1.4. *The degree of a graph curve, when embedded as a union of lines, is equal to the number of projective lines in the curve, or the number of vertices in the corresponding graph.*

Proof. Theorem 7.7 in Hartshorne gives a formula for the degree of an algebraic variety, derived from the Hilbert polynomial (whose leading coefficient is equal to the degree). The formula for arbitrary variety Y, hypersurface H, and intersection $Y \cap H = Z_1, \ldots, Z_s$, irreducible components, is:

$$\sum_{j=1}^{s} i(Y, H; Z_j) \deg Z_j = (\deg Y)(\deg H).$$

In our case, this simplifies to

$$(\# \text{ of points in } Y \cap H) = \deg Y.$$

Every line should intersect a generic hyperplane exactly once. So the degree is the number of lines. $\square$

Proposition 1.5. *The genus of a (connected) graph curve is equal to $\frac{1}{2}|V|$. It is also equal to the dimension of the basis of the cycle space in the graph G, or $g =$ dim $H_1(G, \mathbb{C})$.*

Proof. (Inspired by §4.1.1 in [8]) The easiest way to see this fact is through topological considerations – these graph curves are Riemann surfaces stapled together, so the closed loops raise the genus. Starting with $2n$ lines, meeting in $3n$ points, $2n-1$ points of intersection would have them all connected linearly, with genus 0. Each additional intersection raises the genus by 1, so we have $g = n + 1$.

We will explore another way to see this fact using the canonical bundle of the curve. $\square$

2 Simplicial Cohomology of G

Graph curves are such a potent tool because the properties of the graph translate into properties of the curve. The key to this connection is the isomorphism between the 1-cocycles of the graph and the differential forms on the curve. We can simplify the discussion later by proving that the cycle space of a graph is isomorphic to the cocycle space. The definitions below are based on [7].

Definition 2.1. The *chain complex* associated to a directed graph is the sequence:

$$0 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

When discussing a graph as a simplicial complex, the 0-chains (elements of C_0) are formal sums of vertices, and the 1-chains (elements of C_1) are formal sums of edges. The edges and vertices are clearly bases of their respective spaces.

The map ∂_1 sends an edge e_i oriented from v_i to v_{i+1} into the 0-chain $(v_{i+1} - v_i)$. Because a graph has no higher-dimensional components, the other maps are trivial.

The *first homology group*, denoted $H_1(G)$, is the quotient group $\mathrm{Ker}(\partial_1)/\mathrm{Im}(\partial_2)$. In the case of a graph, $\mathrm{Im}(\partial_2)$ is zero, so $H_1(G) = \mathrm{Ker}(\partial_1)$.

We define the cochain complex dually:

Definition 2.2. The *cochain complex* associated to a directed graph is the sequence:

$$0 \xleftarrow{\delta_2} C_1^* \xleftarrow{\delta_1} C_0^* \xleftarrow{\delta_0} 0$$

The 0-cochains (elements of C_0^*) are functions from the vertices to $\mathbb{C}$, and the 1-cochains (elements of C_1^*) are functions from the edges to $\mathbb{C}$. In other words, $C_n^* = \mathrm{Hom}(C_n, \mathbb{C})$.

Let the functions e_i^* or v_i^* be cochains valued as 1 on e_i and v_i respectively, and 0 elsewhere. The set of such functions are bases for the space of cochains.

The map δ_1 sends a 0-cochain v_i^*, where edges $e_{i_1}, \ldots, e_{i_m}$ are oriented towards v_i and $e_{j_1}, \ldots, e_{j_n}$ are oriented away from it, to the 1-chain $(\sum e_{i_r}^* - \sum e_{j_s}^*)$, where the subtraction is taken componentwise.

The *first cohomology group*, denoted $H^1(G, \mathbb{C})$, is the quotient group $\mathrm{Ker}(\delta_2)/\mathrm{Im}(\delta_1)$. In the case of a graph, $\mathrm{Ker}(\delta_2)$ is the whole space, because the graph contains no 2-faces; so, $H^1(G, \mathbb{C}) = C_1^*/\ \mathrm{Im}(\delta_1)$.

We can get a better understanding of the cochain space by observing that we can give the space an inner product structure.

Definition 2.3. Define the inner product $\langle \cdot, \cdot \rangle$ on the distinguished basis of the 1-cochain space by $\langle e_i^*, e_j^* \rangle = e_i^*(e_j)$, then extend to the rest of the space linearly. Define the inner product analogously for the space of 0-cochains.

One can check that this satisfies the axioms of inner products. Given that $H^1(G, \mathbb{C}) = C_1^* / \operatorname{Im}(\delta_1)$, we will take advantage of the inner product to classify the elements of this group. $\operatorname{Im}(\delta_1)$ are the coboundaries in C_1^*, so we want to identify the subspace orthogonal to the space of coboundaries.

Proposition 2.4. *Let $\phi = \sum a_i e_i$ be a 1-chain, and $\psi = \sum b_i v_i^*$ a 0-cochain. The value of $\langle (\partial \circ \phi)^*, \psi \rangle$, as a product of 0-cochains, is equal to $\langle \phi*, \delta \circ \psi \rangle$, as a product of 1-cochains.*

Proof. It is sufficient to consider this statement on an element of the basis of the 0-cochain space.

$$\langle (\partial \circ \phi)^*, v_i^* \rangle = \langle (\partial \circ \sum a_i e_i)^*, \psi^* \rangle = \langle \sum a_i (v_j^* - v_k^*), v_i^* \rangle = \sum a_{i_r} - \sum a_{j_s},$$

where e_{i_r} are edges oriented towards v_i and e_{j_s} are oriented away. From the other direction,

$$\langle \phi, \delta \circ v_i^* \rangle = \langle \sum a_i e_i^*, \sum e_{i_r}^* - \sum e_{j_s}^*) = \sum a_{i_r} - \sum a_{j_s}.$$

□

Using this fact it becomes evident that the cochains orthogonal to the coboundaries are precisely the duals to the cycles of $H^1(G, \mathbb{C})$. With ϕ a 1-chain, and ψ a 0-cochain,

$$\langle \phi^*, \delta \circ \psi \rangle = 0 \equiv \langle (\partial \circ \phi)^*, \psi \rangle = 0$$

Because ψ could be any 0-cochain, we must require that $\partial \circ \phi = 0$, i.e. ϕ is a 1-cycle. So, in our discussions later in the paper, we are justified in interchanging 1-cycles and 1-cocycles.

3 The Sheaf of Differentials on $C(G)$

The correspondence with the cohomology of the graph expresses itself in the curve through the canonical sheaf.

Definition 3.1 (§II.8 in [6]). Let V be a nonsingular variety over k. We define the canonical sheaf as $\omega_X = \bigwedge^n \Omega_{X/k}$, the n-th exterior power of the sheaf of differentials, where $n = \dim X$.

The key to characterizing the differentials on a graph curve is in understanding the interplay between the global section and its restriction to individual components. Since the genus of a graph curve is always sufficiently high, the canonical sheaf will contain only holomorphic differentials. On the other hand, the restriction to any component $\mathbb{P}^1$ is a 1-form on a genus 0 curve, which is linearly equivalent to $-2P$, so has two poles.

Therefore, given a global section $\omega \in \Omega_{C(G)/k}$, we want the restrictions $\omega|_{\ell_i}$ to glue together in such a way that the poles on adjacent lines cancel each other out. Therefore, the poles can only be on the nodes of intersection; furthermore, the residues at each node, taken on the two intersecting lines, must sum to zero.

Again, keeping in mind the restrictions to each projective line, we apply the residue theorem to conclude that the sum of the residues on any component $\mathbb{P}^1$ of $C(G)$ must be 0.

To summarize, here are the restrictions on a differential $\omega \in \Omega_{C(G)/k}$:

1. $\omega|_{\ell_i}$ has simple poles at the nodes, and nowhere else.

2. $\sum_{\text{poles on } \ell_i} \text{res}_P \omega|_{\ell_i} = 0$

3. If $P = \ell_i \cap \ell_j$, then $\text{res}_P \omega|_{\ell_i} + \text{res}_P \omega|_{\ell_j}$.

A quick calculation of the dimension of differentials shows that we are on the right track. Given three nodes per line, with $2g-2$ lines, we start with $6g-6$ unknowns for the values of residues. Given the relation between the residues on adjacent lines at a given node, for each of $3g-3$ nodes, we impose $3g-3$ restrictions. Finally, the residue theorem point indicates that each line sums to zero, though the relation on one of the lines is redundant, so we pick up another $(2g-2)-1$ restrictions.

$$(6g-6) - (3g-3) - (2g-3) = g = \dim H^0 \omega_{C(G)}$$

The possible values for the residues give us the full space of differentials on the graph curve.

4 The Canonical Embedding of a Graph Curve

The cocycle space and the sheaf of differentials described in the preceding sections turn out to be identified by the following proposition from [3]. This connection will allow us to easily describe the canonical embedding.

Proposition 4.1 (Proposition 1.1 in [3]). *There is a natural isomorphism $H^0\omega_{C(G)} \cong H^1(G, \mathbb{C})$; in particular $g(C) = g(G)$.*

Proof. We consider a holomorphic differential ω on $C(G)$. As described in Section 3, the residues at a given node P on the pair of incident lines add up to zero. So, in G, we only need one value at any given node to identify the 1-form; if the edge corresponding to P, e, is oriented from v_1 to v_2 (corr. to ℓ_1 and ℓ_2) in our chosen orientation, we take the 1-cochain ϕ, such that $\phi(e) = \mathrm{res}_P \omega|_{\ell_2}$.

Furthermore, the residue theorem restriction indicates that the corresponding cochain is, in fact, a cocycle. Using the inner product structure that we explained earlier, the fact that the sum of the residues $\sum_{\text{poles on } \ell_i} \mathrm{res}_P \omega|_{\ell_i} = 0$ is equivalent to saying that $\langle \phi, \delta v_i^* \rangle = 0$, for all $v_i \in V$. In other words, a cochain corresponding to a differential is orthogonal to all coboundaries; therefore, it is in the cohomology group $H^1(G, \mathbb{C})$. $\square$

Having defined the sections of the canonical sheaf, we can discuss the canonical map. Bayer and Eisenbud prove that the canonical series is base point free iff the graph G is 2-connected; this condition implies that the map is well-defined. They also prove that the canonical series is very ample iff the graph G is 3-connected; this implies that the map will in fact be an embedding. (Recall that n-connectivity in a graph is the condition that at least n vertices must be removed to render the graph disconnected.)

We now explicitly describe the canonical embedding of a graph curve. We start by setting $T = \mathrm{Sym}(\mathrm{Coch}(G))$, the symmetric algebra on the ring of cochains. Then, we take $S = \mathrm{Sym}(H^1(G, \mathbb{C}))$, the symmetric algebra on the ring of cocycles. As mentioned before, these can be put in bijection with the chains and cycles of the graph. Clearly, S can be considered a subring of T, since all cocycles are cochains. To simplify matters, T can be thought of as a polynomial ring in $2g-2$ variables ($\mathbb{C}[x_1, \ldots, x_{2g-2}]$), i.e. the number of edges; S can be thought of as the subring generated by the sums of variables corresponding (according to orientation). Let $\mathcal{R}$ = the canonical ring of C.

We are looking for the ideal I which is the kernel of the map $S \to R$. Then the image of the ideal in R will define our canonical curve.

Theorem 4.2 (Proposition 3.1 in [3])**.** *I is the intersection of S and the ideal of T generated by all monomials of the forms xy, where x and y are dual to disjoint edges of G, and xyz where x,y, and z are dual to the edges of a triangle in G.*

Proof. These generators are found by considering the ideal corresponding to an individual line, and then taking the intersection over all of the component lines in the graph curve (the intersection of ideals corresponds to the union of varieties).

The ideal of each line will be the intersection of S, the ring of canonical sections, with the ideal generated by those edges not incident to it. When you intersect all such ideals, you obtain the intersection of S with the ideal generated by products of edges that do not share an incident vertex.

The exception to this is when the graph contanis triangles. Any pair of edges in a triangle share a vertex, so none of them are non-adjacent; therefore, none of these quadratic monomials will be included in the ideal. However, the degree-3 monomial consisting of all three variables contains an edge disjoint from each vertex. If we include these in our list of generators, we have obtained all the elements in the canonical ideal. □

5 Computations of Canonical Ideals

Define the graph $G = (V, E)$ as in Figure 3. Using Macaulay2, we set up the polynomial rings $S = \mathrm{Sym}(H^1(G), \mathbb{C})$ and $T = \mathrm{Sym}(\mathrm{Coch}\ G)$. We define the ideal I, generated by monomials corresponding to triangles and pairs of disjoint edges in the graph. We define the map $\phi : S \to T$, by mapping cocycles to the sum of variables dual to the edges in the cycle (modifying sign according to orientation).

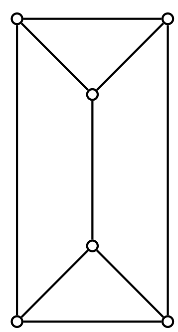

Figure 3: Graph for Canonical Ideal Calculation

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S = ZZ/32003[w..z]
T = ZZ/32003[a..i]
I = ideal(a*b*c,g*h*i,a*d,a*e,a*f,a*g,a*h,a*i,b*f,b*g,b*h,b*i,c*d,
          c*g,c*h,c*i,d*e,d*f,d*h,e*f,e*i,f*g)
phi = map(T, S, {a+c+b,c+e+h+f,h+i+g,b+d+g-e})
```

Following Proposition 3.1 in Bayer-Eisenbud, we obtain the intersection of S with I. The Proposition points to the intersection of $\phi(S)$ with I as the canonical ideal; however, the plurality of variables would only confuse matters. Instead, we look at I's preimage in S.

```
i8 : I2 = preimage(phi,I)
                                2                 2   2          2          2
o8 = ideal (w*y, w*x*z - x z + x*y*z - x*z , x y*z - x*y z + x*y*z )

i9 : mingens I2
o9 = | wy wxz-x2z+xyz-xz2 |
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As shown by the Macaulay2 output, the curve in $\mathbb{P}^3$ is defined by the equations $P = wy$ and $Q = xz(w - x + y - z)$. This curve turns out to be a complete intersection of two surfaces, one with degree 2 and one with degree 3.

Another important graph which is embedded as a complete intersection is the complete bipartite graph $K_{3,3}$. Its canonical ideal is $\langle wx + wy + xz, yz(w - x)\rangle$.

Not all graph curves will be complete intersections in the canonical embedding. One example (which, incidentally, was central to the argument found in [5]) is the Petersen graph, which has an embedding with a minimal set of generators $\langle (w+x)z, (v+w)y, vx - wy + xz, ux + uy - xz, uw - wy + xz, uv + wy + uz\rangle$.

6 Research Directions

1. The gonality of a curve C is defined as the lowest degree of a non-constant rational map from C to the projective line. Equivalently, if C is defined over the field K and $K(C)$ denotes the function field of C, then the gonality is the minimum value taken by the degrees of field extensions $K(C)/K(f)$ of the function field over its subfields generated by single functions f. The gonality is often connected to the Clifford Index, the topic of Bayer and Eisenbud's paper.

   This is the most active area of research in connection with graph curves. While most of the work using graph curves was performed in the late 1980s, the articles [1] and [2] were published since 2009. Ballico gives a slightly modified definition of gonality that makes the gonality of a graph curve always equal to $2g - 2$. Additionally, Lohne, in [9] explores the possibility of connecting an analogue of gonality on a simplicial graph to the gonality of corresponding graph curves.

2. Combinatorics of generators of the canonical ideal. In [4], the authors studied the combinatorics of generators from a sub-trivalent version of graph curves. We found that given a specific algorithm for embedding the curve, the ideal will be generated by products of linear forms of a particular type. Because trivalent graph curves have more inherent symmetry, the generators of ideals may also be combinatorially predictable given the corresponding graph.

   One class of curves which may yield to examination are prism graphs, a sampling of which are pictured in Figure 4. These are nicely characterized and may yield some interesting combinatorics.

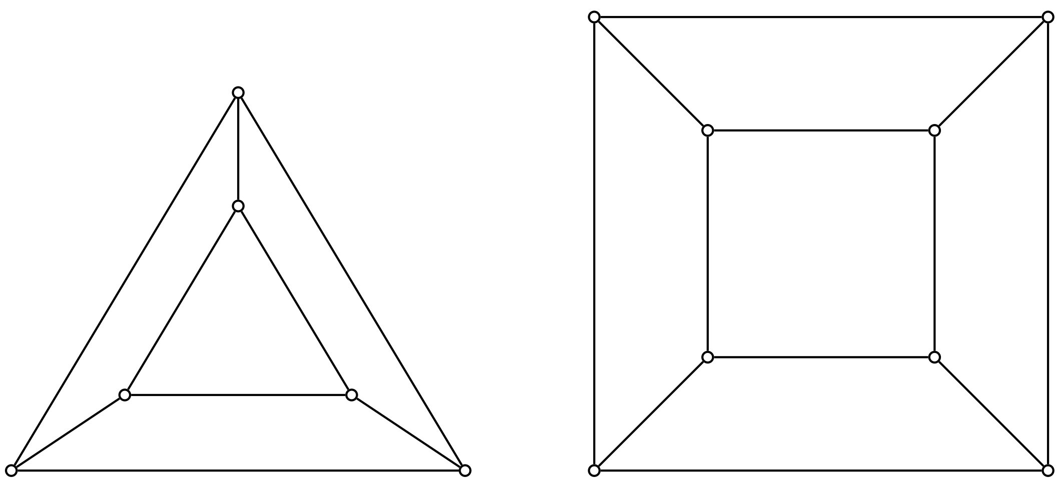

Figure 4: Prism Graphs for $n = 3, 4$

3. The final research direction we will mention is other structures built out of graph curves. In particular, the construction of the *first secant variety* can be carried out by taking any two points in the variety and then taking the projective line that they generate. The union of all such lines is called the first secant variety. In the case of a graph curve, the first secant variety will be a union of 3-planes intersecting in 2-planes. The combinatorics of these objects are interesting, and they also arise in natural problems.

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