OPEN PROBLEM SESSION
ECCO 2012

(1). (Federico Ardila) For a composition \( c = (c_1, \ldots, c_k) \) we are interested in the composition polynomial \( g_c(q) \), which can be given at least three definitions.

(a.) If we write \( t^{c-1} := t_1^{c_1-1} \cdots t_k^{c_k-1} \), where \( t = (t_1, \ldots, t_k) \), then

\[
g_c(q) := \int_1^q \int_q^{t_k} \cdots \int_q^{t_2} t^{c-1} dt_1 \cdots dt_k.
\]

(b.) Let \( \beta_i = c_1 + \cdots + c_i \) for \( i = 0, \ldots, k \). Let \( h(x) = a_0 + a_1 x + \cdots + a_k x^k \) be the polynomial of smallest degree that passes through the \( k+1 \) points \((\beta_i, q^{\beta_i})\) on the curve \( y = q^x \). Here the coefficients \( a_i \) are functions of \( q \). Then \( a_k = (-1)^k g_c(q) \).

(c.) It is the volume of a combinatorially defined polytope, as explained in [1].

Some examples are:
- \( g_{(1,1,1,1)}(q) = \frac{1}{24} (1 - q)^4 \).
- \( g_{(2,2,2,2)}(q) = \frac{1}{384} (1 - q)^4 (1 + q)^4 \).
- \( g_{(1,2,2)}(q) = \frac{1}{120} (1 - q)^3 (8 + 9q + 3q^2) \).
- \( g_{(2,2,1)}(q) = \frac{1}{120} (1 - q)^3 (3 + 9q + 8q^2) \).
- \( g_{(5,3)}(q) = \frac{1}{120} (1 - q)^2 (5 + 10q + 15q^2 + 12q^3 + 9q^4 + 6q^5 + 3q^6) \).

and it is a fact that

\[
g_c(q) = (1 - q)^k f_c(q)
\]

where \( f_c(q) \) is a polynomial with positive coefficients. [1, Theorem 6.5]

Questions:
(I) These polynomials originally arose as volumes of polytopes; why do they also appear in the polynomial interpolation of exponential functions?
(II) Are the coefficients of \( f_c(q) \) unimodal? Are they log-concave?
(III) After suitable rescaling, do the coefficients of \( f_c(q) \) count nice combinatorial objects?

(2). (Criel Merino) Let

\[
M_{r,d} := \{ \text{monomials over } z_1, z_2, \ldots, z_d \text{ of degree } \leq r \}.
\]

A set of monomials \( C_{r,d} \) of degree \( r \) over the variables \( z_1, z_2, \ldots, z_d \) is a covering set for \( M_{r-1,d} \) if any monomial in \( M_{r-1,d} \) is a divisor for some monomial in \( C_{r,d} \).

Now let

\[
f_{r,d} := \min \text{ size of a covering set for } M_{r-1,d}.
\]

Conjecture:
(I) \( f_{r,d} = \#( \text{aperiodic necklaces with } r \text{ black beads and } d - r \text{ white beads}) \).
(II) \( f_{r,d} = f_{d,r} \) [Remark that this is a consequence of the previous conjecture].
(3). (Bernd Sturmfels) Let \( F \) be a family of non-trivial subsets of \([n]\). The collection \( F \) defines a family \( C \) of affine hyperplane arrangements in \( \mathbb{R}^{n-1} \) as follows:

\[
C = \{ \sum_{i \in F} x_i = 0 \} \quad \text{for} \quad F \in \mathcal{F}
\]

**Question:** How many bounded regions does this family have? This may be intractable in general, but an answer for particular families \( F \) would be interesting.

Now, let \( P \) be a poset on \([n]\) and put

\[
\mathcal{L}[P] = \{ \text{linear extensions of } P \}.
\]

**Question:** Determine the kernel of the map

\[
\phi : \mathbb{R}[p_\pi | \pi \in \mathcal{L}[P]] \to \mathbb{R}(x_1, \ldots, x_n)
\]

where \( p_\pi \) is the probability of observing the permutation \( \pi \) in \( \mathcal{L}[P] \) and

\[
\phi(p_\pi) = \prod_{i=1}^{n} \frac{1}{x_{\pi(1)} + \cdots + x_{\pi(i)}}
\]

(4). (Nantel Bergeron) The space \( NC\text{Sym} \) is a subspace of \( k\langle\langle x_1, x_2, \ldots \rangle\rangle \). For \( n \) fixed the following questions are open:

(I) Is \( \langle NC\text{Sym}^+(n) \rangle \) finitely generated?

(II) Is the dimension of the vector space \( k\langle x_1, \ldots, x_n \rangle / \langle NC\text{Sym}^+(n) \rangle \) finite?

(III) What would be the representation theory of \( S_n \) on this quotient?

(IV) Same questions are unsolved for the space \( NC\text{QSym} \).

(5). (Mauricio Velasco) Let

\[
\mathcal{H}_d^n = \{ I \subseteq R = k[x_1, \ldots, x_n] \mid \dim_k(R/I) = d \}
\]

This is the Hilbert scheme on \( d \) points in affine \( n \)-space. Now let

\[
\varphi(d, n) := \sup_I \dim_k(\text{Hom}(I, R/I))
\]

**Question:** What is \( \varphi(3, n) \)?

(6.) (Alejandro Morales) We denote by \( \mathfrak{S}_n \) the group of permutations on \([n] = \{1, 2, \ldots, n\} \).

We write permutations as words \( w = w_1 w_2 \cdots w_n \) where \( w_i \) is the image of \( w \) at \( i \).

We also identify each permutation \( w \) with its permutation matrix, the \( n \times n \) 0-1 matrix with ones in positions \((i, w_i)\). We think of the 1s in a permutation matrix as \( n \) non-attacking rooks on \([n] \times [n] \). Given a subset \( B \) of \([n] \times [n] \) we look at rook placements \( C \) of \( n \) non-attacking rooks on \( B \).

Recall the notion of the strong Bruhat order \( \prec \) on the symmetric group \([2] \text{ Ch. 2}: \) if \( t_{ij} \) is the transposition that switches \( i \) and \( j \), we have as our basic relations that \( u \prec u \cdot t_{ij} \) in the strong Bruhat order when \( \text{inv}(u) + 1 = \text{inv}(u \cdot t_{ij}) \), and we extend by transitivity. Let \([w, w_0]\) denote the interval \( \{ u \mid u \succ w \} \) in the strong Bruhat order where \( w_0 \) is the largest element \( n \cdot n - 1 \ldots 21 \) of this order.
Example 1. If $w = 3412$, then the permutations in $S_4$ that succeed $w$ in the Bruhat order are \{3412, 3421, 4312, 4321\}.

In [4], Sjöstrand gave necessary and sufficient conditions for $[w, w_0]$ to be equal to the set of rook placements of a skew shape associated to $w$. Namely, the left hull $H_L(w)$ of $w$ is the smallest skew shape that covers $w$. Equivalently, $H_L(w)$ is the union over non-inversions $(i, j)$ of $w$ of the rectangles $(k, \ell) \mid w_i \leq k \leq w_j, i \leq \ell \leq j$. See Figure 1 for an example of the left hull of a permutation.

**Theorem 2** ([4, Cor. 3.3]). The Bruhat interval $[w, w_0]$ in $S_n$ equals the set of rook placements in the left hull $H_L(w)$ of $w$ if and only if $w$ avoids the patterns 1324, 24153, 31524, and 426153.

A natural family of diagrams is the collection of Rothe diagrams of permutations, which appear in the study of Schubert calculus. The Rothe diagram $R_w$ of a permutation $w$ is a subset of $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$ whose cardinality is equal to the number of inversions of $w$; it is given by

$$R_w = \{(i, j) \mid 1 \leq i, j \leq n, \; w(i) > j, \; w^{-1}(j) > i\}.$$ 

See Figure 1 for some examples of Rothe diagrams. The following is a special case of two conjectures in [3, Sec. 6].

**Conjecture 3** ([3]). Fix a permutation $w$ in $S_n$. We have that the number of rook placements in the left hull $H_L(w)$ equals the number of rook placements in the Rothe diagram $R_w$ if and only if $w$ avoids the patterns 1324, 24153, 31524, and 426153.

**References**


