Notes were taken by Zvi Rosen. Thanks to Alejandro Morales for providing Figure 2.

## 1. Tuesday, June 19, 2012

**Notation 1.1.** Let CHA denote a combinatorial Hopf algebra.

A combinatorial Hopf Algebra is a *graded* Hopf algebra.

**Definition 1.2.** A graded algebra is a vector space

$$H = \bigoplus_{n \geq 0} H_n, \qquad H_0 = k.$$

This latter condition makes the algebra *connected*. We have a multiplication:

$$m : H \otimes H \to H.$$

Using the formalism of the tensor product presupposes distributive laws, linearity, etc. We also have a unit:

$$u : k \to H, \qquad 1 \mapsto 1.$$

The graded version of these maps are as follows:

$$m_{k,l} : H_k \otimes H_l \to H_{k+l}.$$

$$u : k \to H_0.$$

Maps should be associative and respect the unity, as in Figure 1.

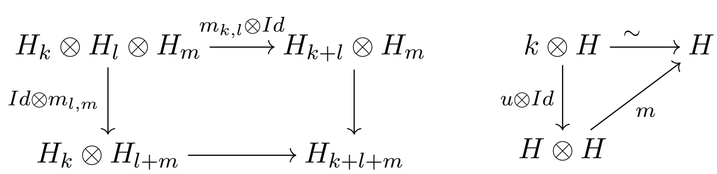



Figure 1. Multiplication and Unit.

**Definition 1.3.** A graded, connected bialgebra also has a co-algebra structure; it is represented by the tuple:

$$\left( H = \bigoplus_{n \geq 0} H_n, m, u, \Delta, \epsilon \right)$$

The last two operations are associated to the coalgebra. We have comultiplication:

$$\Delta : H \to H \otimes H, \qquad a \mapsto \sum a_{(1)} \otimes a_{(2)},$$

as well as a counit:

$$\epsilon : H \to k.$$

The graded versions are as follows:

$$\Delta_{k,l} : H_{k+l} \to H_k \otimes H_l, \qquad \Delta_n = \sum_{k+l=n} \Delta_{k,l}.$$

These operations satisfy the previous diagrams, but with all arrows reversed (so they are coassociative, and respect counity).

For a bialgebra, we require $\Delta, \epsilon$ to be algebra homomorphisms, i.e.

$$\Delta(ab) = \Delta(a)\Delta(b).$$

Multiplying the comultiplied tensors involves a twisting operation $\tau$, which may require some cleverness; for a graphical description, see Figure 2. These operations satisfy the diagram in Figure 3.

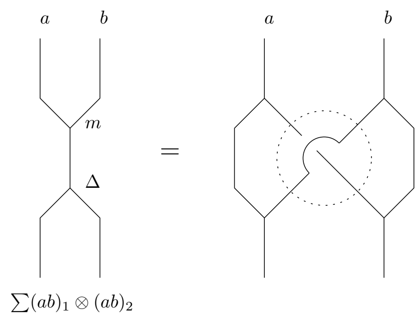



FIGURE 2. $\Delta$ as an Algebra Homomorphism.

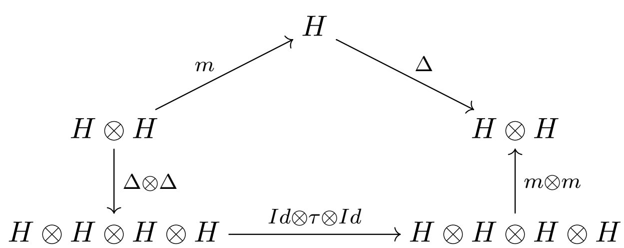



FIGURE 3. $\Delta, m$, and $\tau$.

**Definition 1.4.** A bialgebra $H$ ( = the tuple $\left(H = \bigoplus_{n\geq 0} H_n, m, u, \Delta, \epsilon\right)$) is *Hopf*, if there exists a map $S : H \to H$, such that it satisfies the diagram of Figure 4. $S$ is called the *antipode*.

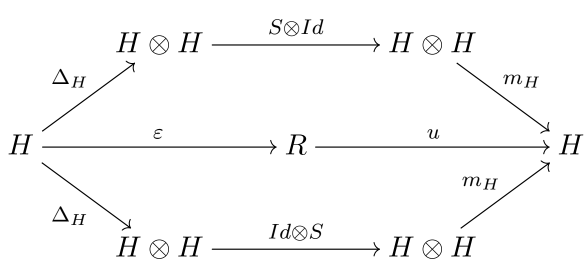



FIGURE 4. The Antipode $S$.

**Remark 1.5.** Why do we care about an antipode? The set

$$\{\varphi : H \to A\} = \mathrm{Hom}_{Alg}(H, A), \qquad A \text{ an algebra}$$

has a multiplication (convolution).

Given $f, g \in \mathrm{Hom}_{Alg}(H, A)$ we convolute them satisfying the diagram in Figure 5.

In fact, $\mathrm{Hom}_{Alg}(H, A)$ is a group under convolution with identity $u_A \circ \epsilon_H$ and inverse

$$f^{(-1)} = f \circ S.$$

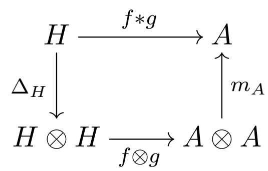



FIGURE 5. Convolution.

**Proposition 1.6.** *The antipode satisfies:*

$$S(ab) = S(b)S(a).$$

*Moreover, if $H$ is commutative or cocommutative (see Figure 6), then*

$$S^2 = Id.$$

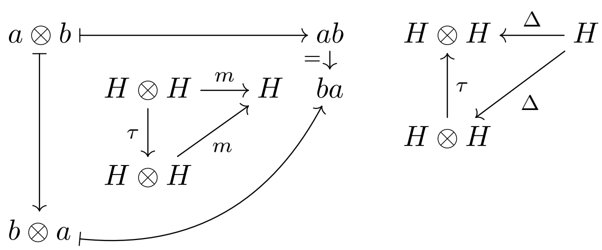



FIGURE 6. Commutativity and Co-commutativity.

**Proposition 1.7.** *For a Graded Hopf Algebra, we have a formula for the antipode. First,*

$$S(1) = 1.$$

*For $h \in H_n, n > 0$,*

$$\Delta(h) = 1 \otimes h + \sum_{a_{(1)} \neq 1, \deg(a_{(1)}) > 0} a_{(1)} \otimes a_{(2)}.$$

$$m \circ (Id \otimes S) \circ \Delta(h) = u \circ \epsilon(h) = 0.$$

$$m \circ (Id \otimes S) \left[ 1 \otimes h + \sum_{\deg(a_{(1)}) > 0} a_{(1)} \otimes a_{(2)} \right].$$

$$= S(h) + \sum_{\deg(a_{(1)}) > 0} a_{(1)} S(a_{(2)}) = 0.$$

*Note that $d(a_{(2)}) < n$. Therefore,*

$$S(h) = -\sum_{\deg(a_{(1)}) > 0} a_{(1)} S(a_{(2)}).$$

*If $H$ is a connected, graded bialgebra, then there is a unique antipode $S$ (given by this formula).*

**Example 1.8** (Symmetric functions)**.** Let $Sym = k[e_1, e_2, \ldots]$, and let $\deg(e_i) = i$. The multiplication is well-known. As a basis, we can take the set of monomials $e_1^{a_1} e_2^{a_2} \cdots$, where finitely many $a$'s are nonzero. For notation, we denote a monomial by a weakly descending sequence of variables in the product, e.g.

$$e_1^2 e_3 e_4^3 = e_4 e_4 e_4 e_3 e_1 e_1 \to e_{444311}.$$

This gives us the set:

$$\{e_\lambda\}_{\lambda \vdash n \geq 0},$$

where $\lambda \vdash n$ indicates that $\lambda$ is a partition of $n$. $\lambda = (\lambda_1, \ldots, \lambda_l)$, where each $\lambda_i > 0$, the sum of the $\lambda$'s is $n$ and, the sequence is weakly decreasing.

When $n = 0$, $\lambda = \emptyset \Rightarrow e_0 = e_\emptyset = 1$. So, $Sym$ is a graded algebra, with

$$Sym = \bigoplus_{n \geq 0} k\{e_\lambda\}_{\lambda \vdash n}.$$

$Sym$ is also a biaglebra. $Sym$ is a free algebra (commutative).

$$\Delta(e_i) = \sum_{k+l=i} e_k \otimes e_l.$$

We need to check that it is coassociative (simply requires viewing a 3-part partition as an iterated 2-part partition in 2 ways).

What is the antipode $S$? Let $S(e_i) = (-1)^i h_i$ (we define $h_i$ in this way).

$$\Delta(e_i) = 1 \otimes e_i + \sum_{k+l=i, k>0} e_k \otimes e_l.$$

$$\Rightarrow (-1)^i h_i = S(e_i) = -\sum_{k=1}^{i} e_k S(e_{i-k}) = -\sum_{k=1}^{i} e_k (-1)^{i-k} h_{i-k}.$$

What do we get when we take $S(h_i)$?

We now commence with four definitions of Combinatorials Hopf Algebras:

**Definition 1.9.** (1) We have a singled out basis such that the structure is positive, i.e. $\{e_\lambda\}$ such that:

$$e_\lambda e_\mu = \sum c^\nu_{\lambda\mu} e_\nu,$$

$$\Delta(e_\nu) = \sum d^\nu_{\lambda\mu} e_\lambda \otimes e_\mu,$$

where the coefficients are positive.

(2) Realization:

$$H \hookrightarrow k[[x_1, x_2, \ldots]] \text{ or } k\langle\langle x_1, x_2, \ldots\rangle\rangle.$$

(3) Representation Theory:

$$H \cong K\left(\bigoplus_{n \geq 0} A_n\right).$$

(4) Via characters:

$$\chi : H \to k, \qquad \text{an algebra homomorphism.}$$

## 2. Wednesday, June 20, 2012

Yesterday, we saw:

(1) Graded Hopf Algebras.
(2) Antipode
(3) Commutative or Co-commutative implies $S^2 = Id$.
(4) $Sym = k[e_1, e_2, \ldots]$, with antipode $h_i = S((-1)^i e_i)$. Because $S$ is an involution, we can also write $Sym = k[h_1, h_2, \ldots]$.
(5) Four definitions of the Combinatorial Hopf Algebra.

Consider $Sym$ via definition (1): Given a singled-out basis with a positive structure, we want to make a rule for the construction of these objects. We want to explain the structure constants with combinatorial rules

$$e_\lambda e_\mu = e_{\lambda\cup\mu}.$$
$$\Delta(e_\lambda) = \sum e_\mu \otimes e_\nu.$$

Consider $Sym$ via definition (2). We truncate the variables, so we work in the "symmetric polynomials" $k[x_1, x_2, \ldots, x_n]$. $S_n$ acts on $R$ in the following way: $\sigma.P(x_1, \ldots, x_n) = P(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.

$$\Lambda_{(n)} = \{P \in R : \sigma.P = P, \ \forall \sigma \in S_n\} = R^{S_n}.$$
$$\Rightarrow \forall P, Q \in \Lambda_{(n)}, PQ \in \Lambda_{(n)}.$$

2.1. **Bases for $\Lambda_{(n)}$.**

(1) **Orbit of a Monomial.** Start with a monomial, then add in all monomials that result from $S_n$ action (without multiplicity.

**Example 2.1.** $x_1^2x_3x_4$. The orbit includes: $x_1^2x_2x_3, x_1^2x_2x_4, x_1x_3^2x_4, \ldots$. The distinguished member of this set is $x_1^2x_2x_3$, since its exponent vector is a weakly decreasing sequence. This is the leading monomial in the lexicographic order.

We define
$$m_\lambda = \sum_{x^\alpha \text{ in orbit of } x^\lambda} x^\alpha.$$
We can project elements of $\Lambda_{(n+1)} \to \Lambda_{(n)}$, with
$$m_\lambda \mapsto \begin{cases} m_\lambda, & \ell(\lambda) \leq n \\ 0, & \text{otherwise.} \end{cases}$$
Taking the inverse limit of this sequence of projections, we obtain:
$$Sym = \varprojlim \Lambda_{(n)} \subset k[[x_1, x_2, \ldots]].$$
The basis is $\{m_\lambda\}_{\lambda\vdash m\geq 0}$, where, as above,
$$m_\lambda = \sum_{x^\alpha \text{ in orbit of } x^\lambda} x^\alpha.$$

(2) **Elementary Symmetric Functions.**

**Theorem 2.2** (Newton)**.**
$$\Lambda_{(n)} = k[e_1, e_2, \ldots, e_n],$$
*where $e_i$ is defined as follows:*
$$\prod_{i=1}^{n}(1 + x_i t) = \sum_{i=0}^{m} e_i t^i.$$
Again, we project elements of $\Lambda_{(n+1)} \to \Lambda_{(n)}$, with
$$e_i \mapsto \begin{cases} e_i, & i \leq n \\ 0, & \text{otherwise.} \end{cases}$$
Taking the inverse limit, we have another basis for $Sym$.

**Remark 2.3.** Presentation of $Sym \subseteq k[[x_1, x_2, \ldots]]$ (Introduction of the $m$ basis).

$$Sym \to \Lambda_{(n)} \subseteq k[x_1, \ldots, x_n].$$

$$\langle \Lambda^+_{(n)} \rangle = \langle f \in \Lambda_{(n)} : f(0, 0, \ldots, 0) = 0 \rangle.$$

$$= \langle e_1, e_2, \ldots, e_n \rangle = \langle h_1, h_2, \ldots, h_n \rangle.$$

$$\dim_k(k[x_1, x_2, \ldots, x_n]/\langle h_1, \ldots, h_n \rangle) = n!.$$

**Remark 2.4.** When we consider the realization $Sym \subseteq k[[x_1, x_2, \ldots]]$, the multiplication is clear – just the usual multiplication of series. Comultiplication, is less obvious.

$$f \in Sym, f = f(x_1, x_2, \ldots) \Rightarrow f(Y + Z) = f(y_1, y_2, \ldots, z_1, z_2, \ldots) = \sum f^{(1)}(Y) f^{(2)}(Z).$$

So, we define the comultiplication:

$$\Delta(f) = \sum f^{(1)} \otimes f^{(2)}.$$

In that spirit, for

$$e_m = \sum_{1 \leq i_1 < i_2 < \cdots < i_m} x_{i_1} \cdots x_{1_m},$$

we have:

$$e_m(y_1, y_2, \ldots, z_1, z_2, \ldots) = \sum_{\substack{1 \leq i_1 < \cdots < i_l \\ 1 \leq j_1 < \cdots < j_{m-l}}} y_{i_1} \cdots y_{i_l} z_{j_1} \cdots z_{j_{m-l}}.$$

$$= \sum_{l=0}^{m} e_l(Y) e_{m-l}(Z).$$

Therefore, we define the comultiplication:

$$\Delta(e_m) = \sum_{l=0}^{m} e_l \otimes e_{m-l}.$$

Now , we consider $Sym$ via definition (3), namely representation. Our claim is that $Sym$ and the representation of $S_n$ are linked.

### 2.2. **Crash Course in Representation Theory.**

**Definition 2.5.** Let $G$ be a finite group. A *representation* is a map $\varphi : G \to GL(V)$, where $V$ is a vector space of dimension $d \Rightarrow GL(V) \cong GL_d(k)$.

**Definition 2.6.** A subspace $W \subset V$ is invariant if:

$$W \subseteq V \text{ s.t. } \forall g \in G, \varphi(g)(w) \in W, \forall w \in W.$$

In character 0, we can decompose $V$ into invariant subspaces

$$V = W \oplus W'$$

$$\Rightarrow \varphi(g) = \varphi|_W(g) \oplus \varphi|'_W(g).$$

**Theorem 2.7.**
- *Every representation decomposes into irreducible representations, i.e. there is no $W \subset V \neq 0$ or $V$ invariant under the group action.*
- *There are finitely many irreducible representations (up to isomorphism) in bijection with the conjugacy classes of $G$.*

For $S_n$, the conjugacy classes are in bijection with $\lambda \vdash n$ (partitions of $n$).

Representation of "Tower" of $S_n$, i.e. $\bigoplus_{n\geq 0} S_n$.
$\{$Irreducible Representations of $\bigoplus S_n\} \longleftrightarrow \lambda \vdash n \geq 0$.
$\Rightarrow k(\bigoplus_{n\geq 0} S_n) = k\{\text{Irr. Reps}\} \cong Sym$, as a graded vector space.

Operations on $k(\bigoplus_{n\geq 0} S_n)$: Let $V$ be any representation of $S_n$.

$$V = \sum c_\lambda X^\lambda,$$

where $X^\lambda$ is the basis of the irreducible representations, and $c_\lambda \in \mathbb{Z}_{\geq 0}$.

Let $V$ be a representation of $S_n$ and $W$ be a representation of $S_m$; then, $V * W$ is a representation of $S_{n+m}$.

Let $H$ be a subgroup of $G$, and $V$ a representation of $H$. Then,

$$Ind_H^G V = \text{“}V \otimes_H kG\text{”}.$$

$$V * W = Ind_{S_n \times S_m}^{S_{m+n}} V \otimes W.$$

Again, supposing $H$ is a subgroup of $G$ with $W$ a representation of $G$. Then

$$Res_H^G W = W|_H.$$

If $V$ is a representation of $S_n$, then

$$\Delta(V) = \sum_{l=0}^{n} Res_{S_l \times S_{n-l}}^{S_n} V = \sum V^{(1)} \otimes V^{(2)}.$$

The Mackey Formula gives us a relation that corresponds to:

$$\Delta(V * W) = \Delta(V) * \Delta(W).$$

So $K(\bigoplus_{n\geq 0} S_n)$ is a Hopf Algebra. Specifically,

$$K(\bigoplus_{n\geq 0} S_n) \xrightarrow{\sim} Sym.$$

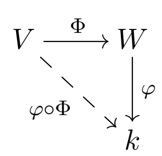
with the special basis:

$$X^\lambda \longrightarrow s_\lambda, \qquad \text{the Schur function.}$$

## 3. Thursday, June 21, 2012

3.1. **Duality.** Given a graded Hopf algebra $H = \bigoplus_{n\geq 0} H_n$, we have a graded dual $H^* = \bigoplus_{n\geq 0} H_n^*$ where $H_n^* = \text{Hom}_k(H_n, k)$.

This duality is explained in the diagram in Figure 3.1.




$$\Phi^* : W^* \longrightarrow V^*.$$

Figure 7. Duality of Algebras.

$H^*$ is also a Hopf algebra, provided that the $H_n$ are finite-dimensional.

(1) For multiplication on $H^*$, we can use $\Delta$ on $H$:

$$f * g = m_k(g \otimes f) \circ \Delta_H,$$

satisfying the diagram in Figure 8.

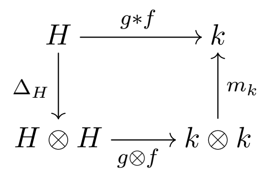



Figure 8. Multiplication on the Dual.

(2) For comultiplication $H^* \longrightarrow H^* \otimes H^*$, we use $m_H$.

$$\Delta(f) = \varphi^{-1} \circ m_H^*(f).$$

See Figure 9 for a description.

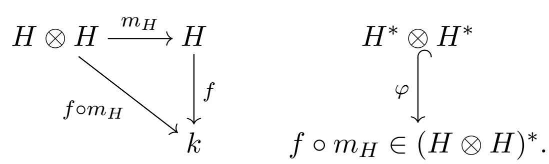



Figure 9. Comultiplication on the Dual.

(3) The unit is obtained from the unit $u : k \longrightarrow H$:

$$\Rightarrow H^* \xrightarrow{u^*} k^* \xrightarrow{\sim} k.$$

(4) Similarly, the counit is obtained from the counit $\epsilon : H \to k$:

$$\Rightarrow k \xrightarrow{\sim} k^* \xrightarrow{\epsilon^*} H^*.$$

Given that the dual to the Hopf algebra is a Hopf algebra, what is $Sym^*$?
We have several bases of $Sym$: $\{e_\lambda\}$, $\{h_\lambda\}$, $\{m_\lambda\}$, $\{s_\lambda\}$.
Let us consider $h_\lambda^* : Sym \to k$, which maps

$$h_\mu \mapsto \begin{cases} 1, & \mu = \lambda \\ 0, & \text{otherwise.} \end{cases}$$

An alternative definition looks at the inner product that gives $\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}$. Then,

$$h_\lambda := \langle - - -, m_\lambda \rangle.$$

When we dualize $e_\lambda$, we get an ugly basis $f_\lambda$, so we ignore it. On the other hand, $h_\lambda$ and $m_\lambda$ dualize to each other, and $s_\lambda$ dualizes to itself. This condition is so special in Hopf algebras that it uniquely characterizes tensor powers of $Sym$ (Zelevinsky).

3.2. **NSym.** Let us describe a new Hopf algebra.

**Definition 3.1.** Let $NSym = k\langle H_1, H_2, \ldots \rangle$ be the free associative algebra on the variables $H_i$, where the degree of $H_i = i$.

Monomials are words in the $H_i$'s:

$$H_\alpha = H_{\alpha_1} H_{\alpha_2} \cdots H_{\alpha_l}.$$

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l), \alpha_i > 0, \alpha_1 + \alpha_2 + \cdots + \alpha_l = m = |\alpha| \Rightarrow \alpha \models m.$$

A basis of $NSym$ is $\{H_\alpha\}_{\alpha\models m\geq 0}$. This is a bialgebra with comultiplication:

$$\Delta(H_m) = \sum_{i=0}^{m} H_i \otimes H_{m-i}.$$

Therefore, $NSym$ is Hopf, with $S(H_i) = (-1)^i E_i$.

Now we consider the dual of $NSym$, i.e. $Nsym^*$. In $NSym$ the multiplication is non-commutative, while the comultiplication is cocommutative; therefore, in the dual $NSym^*$, the multiplication will be commutative, while the comultiplication will be non-cocommutative.

$$NSym^* \tilde{\to} QSym.$$
$$H_\alpha^* \mapsto M_\alpha.$$

$QSym$ is the ring of Quasi-symmetric functions (given by a realization):

$$QSym \subseteq k[[x_1, x_2, \ldots]].$$
$$M_\alpha(x_1, x_2, \ldots) := \sum_{i_1<i_2<\cdots<i_l} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_l}^{\alpha_l}.$$

**Remark 3.2.** The action of $S_n$ on $k[x_1, x_2, \ldots, x_n]$:

$$s_i * x^\alpha = \begin{cases} x^\alpha, & \alpha_i \neq 0 \text{ and } \alpha_i \neq 0 \\ s_i(x^\alpha), & \text{otherwise.} \end{cases}$$
$$\Rightarrow \sigma * (fg) \neq (\sigma * f)(\sigma * g).$$

You can check that $M_\alpha = \sum x^\beta$, where $\beta$ runs over the orbit of the polynomial $x^a lpha$ under the $(*)$.

$$\Rightarrow QSym_{(n)} = k[x_1, x_2, \ldots]^{S_n(*)}, \qquad QSym = \lim_{\longleftarrow} QSym_{(n)}.$$

In $QSym$,

$$M_\alpha M_\beta = \sum_{\gamma\in\alpha\widetilde{Ш}\beta} M_\gamma.$$

This $\widetilde{Ш}$ is a quasi-shuffle, where you can intermix the two words, or superimpose letters from the two words.

**Example 3.3.** The normal shuffle Ш:

$$(1)Ш(2,1) = 121 + 211 + 211.$$

The quasi-shuffle $\widetilde{Ш}$

$$(1)\widetilde{Ш}(2,1) = 121 + 211 + 211 + 31 + 22.$$

For our comultiplication, we take $\Delta(M_\alpha) = \sum_{\alpha=\beta\cdot\gamma} M_\beta \otimes M_\gamma$.

Working with

$$QSym \subseteq k[[x_1, x_2, \ldots]], \qquad QSym_{(n)} \subseteq k[x_1, \ldots, x_n],$$

we may look at

$$\dim(k[x_1, \ldots, x_n]/\langle QSym_{(n)}^+\rangle) = \dim(TL_n),$$

a very surprising fact, where $TL_n$ = Temperley-Leib Algebra $\cong kS_n/(\text{ker action of } *)$.

**Question 3.4.** Given an algebra $A_n$ obtained by generators acting faithfully on $k[x_1, x_2, \ldots, x_n]$, define

$$k[x_1, \ldots, x_n]^{A_n} = \{p \in k[x_1, \ldots, x_n] : g_i P = P \ \forall g_i \text{ generator}\}.$$

When do we have

$$\dim(k[x_1, \ldots, x_n]/\langle k[x_1, \ldots, x_n]^{A_n^+}\rangle) = \dim A_n?$$

Type C Hopf Algebra:

$$NSym \cong K\left(\bigoplus_{n\geq 0} H_n(o)\right) \text{ (Projective representation of } H_n(o)\text{).}$$

$$QSym \cong G\left(\bigoplus_{n\geq 0} H_n(o)\right) \text{ (Finitely generated module of } H_n(o)\text{).}$$

3.3. **NCSym.** We now describe another Combinatorial Hopf Algebra.

$NCSym \subseteq k\langle\langle x_1, x_2, \ldots\rangle\rangle$, the non-commutative $k$-algebra of series. Specifically,

$$NCSym = \varprojlim k\langle x_1, x_2, \ldots, x_n\rangle^{S_n}.$$

$NCSym$ has the basis $M_A = \sum w$, where $w$ is in the orbit of a word $w(A)$. We write $w = x_{i_1} x_{i_2} \cdots x_{i_m}$.

$w$ is also a function: $[1, 2, \ldots, m] \to \{1, 2, \ldots\}$.

$$\nabla(w) = \{w^{-1}(i) : i \in \{1, 2, \ldots\}\} \setminus \emptyset.$$

This is a set partition of $\{1, 2, \ldots, m\}$; therefore, orbits are in 1-to-1 correspondence with $A \vdash \{1, 2, \ldots, m\}, m \geq 0$. (Note: these are set partitions of the set of integers).

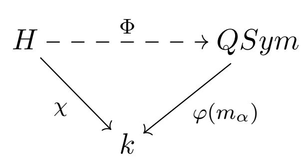
A basis of $NCSym$ is given by

$$\{M_A\}_{A\vdash[m], m\geq 0}.$$

**Question 3.5.** What is the dimension of

$$k\langle x_1, x_2, \ldots, x_n\rangle/\langle NCSym^+\rangle?$$

Now, we examine a Hopf Algebra defined by its character. See Figure 10 for a description.




$$\varphi(m_\alpha) = m_\alpha(1, 0, 0, \ldots)$$
$$= \begin{cases} 1 & \alpha = (n), n \geq 0. \\ 0 & \text{otherwise} \end{cases}$$

Figure 10. QSym, defined by Character.

## 4. Friday, June 22, 2012

A graded Hopf algebra can also be thought of as a set of vector spaces spanned by constructing combinatorial objects, i.e.

$$H = \bigoplus_{n\geq 0} H_n = \bigoplus_{n\geq 0} H[n].$$

Then, we can define an exponential generating function:

$$H(z) = \sum_{n\geq 0} \dim H[n] \frac{z^n}{n!}.$$

(1) Multiplication: $H[n] \otimes H[m] \to H[n+m]$, sending $a \otimes b \mapsto a * b \uparrow^n$, where $\uparrow^n$ sends a combinatorial object on $\{1,\dots,m\}$ to an object on $\{n+1,\dots,n+m\}$.
(2) Comultiplication: $\Delta : H[n] \to \bigoplus_{k+l} H[k] \otimes H[\ell]$, sending $a \mapsto \sum_{S\subseteq\{1,\dots,n\}} st(a|_S) \otimes st(b|_S)$.
 Here $st$ is a function sending a set $S$ to the set of integers from 1 to $|S|$.
(3) Antipode: $S(h) = -\sum_{h_{(1)}\neq 1} h_{(1)} S(h_{(2)}) = \sum_{h_{(1)},h_{(2)}} (-1)^{\Box} h_{(1)} h_{(2)}$.

**Definition 4.1.** Given a finite set $S$, a species gives a graded vector space of structures. For example,
$S \longrightarrow H[S]$, a finite-dimensional vector space of a certain construction on $S$.
$S \longrightarrow G[S]$, the space of graphs on $S$.
Not only do we know how to construct $H[S]$, but we have a natural transformation that takes

$$[\varphi : S \xrightarrow{\sim} T] \longrightarrow [H[\varphi] : H[S] \xrightarrow{\sim} H[T].$$

In other words, a species is a functor from $FiniteSets$ to $VectorSpaces$.

**Example 4.2.** (1) $E$, the "Exp" species, given by $E[S] = k\{S\}$.
(2) $\Pi$, "Set Partitions", is given by $\Pi[S]$ = the span of set partitions on $S$. For example,

$$\{a,b,c\} \longrightarrow \Pi[\{a,b,c\}] \text{ with basis } \{\{a,b,c\}\}, \{\{a\},\{b,c\}\}, \dots, \{\{a\},\{b\},\{c\}\}.$$

4.1. **Hopf Monoids.** Consider graded vector spaces:

$$H = \bigoplus_{n\geq 0} H_n, T = \bigoplus_{n\geq 0} T_n.$$

We have the tensor product:

$$H \otimes T = \bigoplus_{d\geq 0} \left( \bigoplus_{n+m=d} H_n \otimes H_m \right).$$

Multiplication: $m : H \otimes H \to H$.
Graded Multiplication: $m_{n,m} : H_n \otimes H_m \to H_{n+m}$.
We similarly define the tensor product of species:

**Definition 4.3.** Let $A$ and $B$ be two species. Then, we write $A \bullet B$ is the species such that

$$A \bullet B[S] = \bigoplus_{I+J=S} A[I] \otimes B[J].$$

where " + " is the disjoint union.

**Definition 4.4.** We define the multiplication map $m : H \bullet H \to H$, by taking for all $I + J = S$,

$$m_{I,J} : H[I] \otimes H[J] \longrightarrow H[S], \text{ with } a \otimes b \mapsto a * b.$$

**Definition 4.5.** We define the Hopf Monoid:

$$(H, m, u, \Delta, \varepsilon, S).$$

where $H$ is a species, $m$ is a multiplication map from $H \bullet H \to H$ (satisfying figure 11), $u$ is the unit mapping $\mathbf{1} \to H$, where:

$$\mathbf{1}[S] = \begin{cases} k & S = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

The comultiplication $\Delta : H \to H \bullet H$ is defined by

$$\Delta_S = \sum_{I+J=S} \Delta_{I,J}, \text{ where.}$$

$$\Delta_{I,J} : H[S] \to H[I] \otimes H[J], \text{ sending } a \mapsto a|_I \otimes a|_J.$$

$m$ and $u$ are associative and unital. $\Delta$ and $\varepsilon$ are coassociative and counital. Compatibility of $\Delta$ on $m$ is given by the diagram in Figure 12.

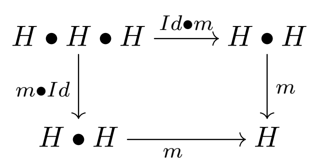



Figure 11. Associativity of the Monoid.

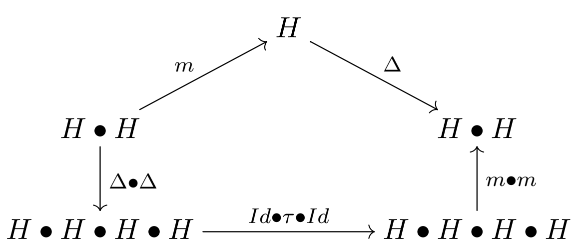



Figure 12. Compatibility of $\Delta$ and $m$.

**Example 4.6.** Consider the species of linear orderings $L[S]$. For any $I + J = S$,

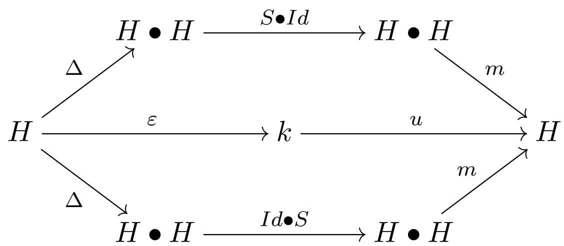
$$m_{I,J} : H[I] \otimes H[J] \to H[I + J],$$

sending $a \otimes b \mapsto a \bullet b$, the concatenation product. For instance, if $I = \{2, 4, 5, 7\}$ and $J = \{1, 3, 6\}$, with order $a = 5274$ and $b = 136$, then $a \bullet b = 5274136$.

$$\Delta_{I,J} : H[S] \to H[I] \otimes H[J],$$

sending $a$ to its restrictions $a|_I \otimes a|_J$.For instance, if $I = \{1, 5\}$ and $J = \{2, 3, 4\}$, with order $a = 43125$, then $\Delta_{I,J}(a) = 15 \otimes 432$.

Furthermore, the species respects all diagrams, so $L$ is a Hopf Monoid.




and for all $I + J = S$,

$$\begin{array}{ccc} H[I] \otimes H[J] & \xrightarrow{Id \otimes S} & H[I] \otimes H[J] \\ \Delta_{I,J} \uparrow & & \downarrow m_{IJ} \\ H[S] & \xrightarrow{u \circ \varepsilon} & H[S] \end{array}$$

$$\text{where } u \circ \varepsilon = \begin{cases} 0 & S \neq \varnothing \\ 1 & S = \varnothing \end{cases}.$$

FIGURE 13. The Antipode $S$.

The graded antipode requirement is found in Figure 13. The antipode map is defined for each set $S$.

Take $a \in H[S]$ such that $S \neq \emptyset$. We have a recursive formula:

$$\mathcal{S}(a) = - \sum_{I+J=S} a|_I \mathcal{S}(a|_J).$$

$$\mathcal{S} : H \to H, \text{ and } \mathcal{S}_S : H[S] \to H[S].$$

$a \in H[\emptyset] = k$ and $S_\emptyset(1) = 1$.

$$H \xrightarrow{K} \bigoplus_{n \geq 0} H[n].$$

$$H \xrightarrow{\bar{K}} \bigoplus_{n \geq 0} H[n]_{S_n}.$$

This latter species is associated to the unlabeled case – since structures are invariant under label-shuffles:

$$H[n]_{S_n} = H[n] / \langle x - H[\sigma](x) : \forall x \in H[n], \forall \sigma : [n] \xrightarrow{\sim} [n] \rangle.$$

Passing to Generating Functions:

$$\bigoplus_{n \geq 0} H[n] \longrightarrow H(z) = \sum_{n \geq 0} \dim(H[n]) \frac{z^n}{n!}.$$

$$\bigoplus_{n \geq 0} H[n]_{S_n} \longrightarrow H(z) = \sum_{n \geq 0} \dim(H[n]_{S_n}) z^n.$$

The Hopf Algebras that we discussed can be related to species: for example, $QSym$ can be obtained via $\mathcal{L} \circ \mathcal{E}^+$.