1. Antipode

Let $H$ be a Hopf algebra and let $S$ be its antipode.

(1). Show that

\[ S(gh) = S(h)S(g) \]

for $g, h \in S$.

(2). Show that if $H$ is commutative or cocommutative, then $S^2 = I_H$.

(3). Let $F = \text{Hom}_{\text{Alg}}(H, k)$ be the set of algebra morphisms from $H$ to the ground field $k$. Show:

(a) $F$ is a group under the convolution product $\ast$ where

\[ g \ast f := m_k(g \otimes f)\Delta_H \]

(b) For $f \in F$ we have

\[ f \circ S = f^{-1} \]

2. Homogeneous and Elementary Symmetric Functions

(1). Given that

\[ h_n = S((-1)^ne_i) = -\sum_{i=1}^{n} (-1)^i h_{n-i}e_i \]

write $h_i$ in terms of the $e_i$’s for $i = 1, \ldots, n$.

(2). Use the identity

\[ \sum_{i=0}^{m} e_i t^i = \prod_{i=1}^{m} (1 + tx_i) \]

to write an expression for $e_i(x_1, \ldots, x_m)$.

- Give an expression for $h_i(x_1, \ldots, x_m)$ using (1) and (2). [Hint: Guess and prove].
- Define the algebra map

\[ \omega : \text{Sym} \to \text{Sym} \]

such that $\omega(e_i) = h_i$.

- Prove that $\omega$ is an involution.
- Conclude that $\text{Sym} \cong \mathbb{Z}[h_1, h_2, \ldots]$.
– Compute $S(h_i)$.
– Compute $\Delta(h_i)$.

(5). Show the following:
   (a) $h_k(x_1, \ldots, x_n) = h_k(x_2, \ldots, x_n) + x_1h_{k-1}(x_1, \ldots, x_n)$.
   (b) $h_k(x_1, \ldots, x_n) \in \langle \text{Sym}_n^+ \rangle$.
   (c) Using the order $x_1 > \cdots > x_n$ show that $\text{LM}(h_k(x_k, \ldots, x_n)) = x_k^k$ where $\text{LM}(f)$ denotes the leading monomial of the polynomial $f$.

   [Note: Given two monomials $x_1^{a_1}x_2^{a_2} \cdots x_i^{a_i}$ and $x_1^{b_1}x_2^{b_2} \cdots x_k^{b_k}$, we say that $x_1^{a_1}x_2^{a_2} \cdots x_i^{a_i} \geq x_1^{b_1}x_2^{b_2} \cdots x_k^{b_k}$ whenever $(a_1, \ldots, a_i) \geq_{\text{lex}} (b_1, \ldots, b_k)$. The leading term $\text{LM}(f)$ is the maximum of the monomials in $f$ under the order $\geq$.]

   (⋆) Show that the set $\{h_i(x_1, \ldots, x_n)\}_{i \in [n]}$ is a Groebner basis for $\langle \text{Sym}_n^+ \rangle$.

   Conclude that the dimension of the vector space $\mathbb{Z}[h_1, \ldots, h_k]/\langle \text{Sym}_n^+ \rangle$ is $n!$. 