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1. Tuesday, June 12, 2012

Combinatorics is the study of finite structures that combine via a finite set of rules. Algebraic combinatorics uses algebraic methods to help you solve counting problems. Often algebraic problems are aided by combinatorial tools; combinatorics thus becomes quite interdisciplinary.

Analytic combinatorics uses methods from mathematical analysis, like complex and asymptotic analysis, to analyze a class of objects. Instead of counting, we can get a sense for their rate of growth. Source: *Analytic Combinatorics*, free online at `http://algo.inria.fr/flajolet/Publications/book.pdf`.

Goal 1.1. Use methods from complex and asymptotic analysis to study qualitative properties of finite structures.

Example 1.2. Let S be a set of objects labeled by integers $\{1, \dots, n\}$; how many ways can they form a line?

Answer: $n!$. How fast does this quantity grow? How can we formalize this notion of quickness of growth? We take a set of functions that we will use as yardsticks, e.g. line, exponential, etc.; we will see what function approximates our sequence.

Stirling's formula tells us:

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}.$$

where $a_n \sim b_n$ implies that $\lim_{n\to\infty} \frac{a}{b} \to 1$.

This formula could only arrive via analysis; indeed Stirling's formula appeared in the decades after Newton & Leibniz.

Almost instantly, we know that $(100)!$ has 158 digits. Similarly, $(1000)!$ has 10^{2568} digits.

Example 1.3. Let an alternating permutation $\sigma \in S_n$ be a permutation such that $\sigma(1) < \sigma(2) > \sigma(3) < \sigma(4) > \cdots > \sigma(n)$; by definition, n must be odd.

Let T_n be the number of alternating permutations in S_n.

Listing the first few terms of this sequence we have:

n	T_n	$n!$
1	1	1
3	2	2
5	16	120
7	272	5040
9	7936	$362,880$

It appears that $n!$ grows "much faster" than T_n. How do we quantify this notion of "much faster"?

(1) Find a formula to approximate T_n.
(2) Compare it to $n!$ or the Stirling formula, i.e. we look at the ratio $T_n/n!$ as $n \to \infty$. If the limit is 1, then the growth is as fast, if the limit is 0, then we say it is "much slower".

We want to find a formula that approximates T_n; in 1881, Desire Andre found that the coefficient of $\frac{z^n}{n!}$ in the function $\tan(z)$ is precisely T_n.

Therefore, $\tan(z)$ is the exponential generating function (e.g.f) for T_n.

We now describe a bijection between these permutations and the set of decreasing binary trees, i.e. a rooted binary tree where the value of a child is always less than the value of a parent.

For example, taking the permutation

$$\binom{123456789}{572836491},$$

we translate it into a decreasing binary tree by starting with the largest number, and letting the left child and the right child be the largest number to its left and its largest child to the right, then iterate.

So, an alternating permutation looks like $(\sigma_L, \max, \sigma_R)$.

Let τ be the class of alternating permutations. Then,

$$\tau = (1) \cup (\tau, \max, \tau).$$

$$\Rightarrow T(z) = z + \int_0^z T(\omega)^2 d\omega.$$

$$\Rightarrow \frac{dT(z)}{dz} = 1 + T(z)^2, \quad T(0) = 0,$$

a differential equation that we solve and find $T(z) = \tan(z)$.

Remark 1.4. What we have learned:

(1) A simple algorithm to find T_n.
(2) g.f admits an explicit expression in terms of a well known mathematical object, i.e. $\tan(z)$.
(3) the probability of randomly obtaining an alternating permutation: $\frac{T(n)}{n!}$ is the coefficient of z^nin Taylor expansion of $\tan(z)$.

There is an amazing relationship between the growth of the coefficients of a counting sequence and the singularities of the generating function when viewed as an analytic function $\mathbb{C} \to \mathbb{C}$.

Example 1.5. The sequence $1, 1, 1, 1, 1, \ldots$ are the coefficients of the generating function:

$$\sum_{n\geq 0} z^n \text{ ``}=\text{''} \frac{1}{1-z}, \quad |z| < 1,$$

with singularity at $z = 1$.

Example 1.6. The sequence $1, 2, 4, 8, 16, \ldots$ are the coefficients of the generating function:

$$\sum_{n\geq 0} 2^n z^n \text{ ``}=\text{''} \frac{1}{1-2z}, \quad |z| < \frac{1}{2},$$

with singularity at $z = 1/2$.

Based on this, the singularities give us information about the growth of the sequence. Therefore, in Example 1.3, we want to examine the singularities of $\tan(z)$ near zero. The closest singularities are $\pm\frac{\pi}{2}$.

Let us approximate $\tan(z)$ near $\pm\frac{\pi}{2}$, we have:

$$\tan(z) \sim \frac{8z}{\pi^2 - 4z^2}.$$

Extracting the coefficient of z^n, we obtain:

$$\frac{T_n}{n!} \sim 2\left(\frac{2}{\pi}\right)^{n+1} \to 0.$$

Therefore, we can say with more rigor that T_n grows much slower than $n!$.

Remark 1.7. Here, we used only real singularities. However, in general, we must study complex values of z. It is singularities in the complex plane that matter and complex analysis that is needed to draw conclusion regarding the asymptotic behavior of the counting sequence.

Example 1.8. Let c_n = the number of *all* unlabeled ordered binary trees with n internal nodes.

n	c_n
0	1
1	1
2	2
3	5
4	14
5	42

You may recognize these as the *Catalan numbers.* We can obtain this via the symbolic method as well:

$$\begin{aligned} C &= (C, \cdot, C). \\ \Rightarrow C(z) &= 1 + zC(z)^2. \\ \Rightarrow C(z) &= \frac{1-\sqrt{1-4z}}{2z}. \\ &= \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} z^n. \end{aligned}$$

As this is the ordinary generating function, we are concerned with the coefficient of z^n, which gives us the formula:

$$c_n = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{n+1}\frac{(2n)!}{n!n!}.$$

We can approximate this last expression using Stirling's formula:

$$c_n \sim c_n^* = \frac{4^n}{\sqrt{\pi n^3}}.$$

Here the growth is related to a non-pole singularity, i.e. $z = \frac{1}{4}$.

2. Wednesday, June 13, 2012 / Symbolic Method

(Note that all topics discussed today deal with unlabeled structures.)

There are many set-theoretic operations that have a nice translation into generating functions. Today, we will discuss:

- Admissible constructions on combinatorial classes
- Symbolic Method $\rightarrow$ Constructible combinatorial classes $\rightarrow$ O.G.F.

Definition 2.1. A Combinatorial Class is a finite or enumerable set with a size function, such that:

(1) The size of any element ≥ 0.
(2) The number of elements of a given size is finite.

A_n counts number of elements of size n.

Definition 2.2. An ordinary generating function

$$A(z) = \sum_{n\geq 0}^{\mu} A_n z^n,$$

where A is a combinatorial class. Equivalently,

$$A(z) = \sum_{\alpha\in A} z^{|\alpha|}.$$

Admissible Constructions:

m-ary construction: Associates to any collection of m combinatorial classes $B^{(1)}, \ldots, B^{(m)}$, a new class $A = \Phi(B^{(1)}, \ldots, B^{(m)})$. It is admissible iff A_n depends only on $(B_n^{(1)})_{n\geq 1}, \ldots, (B_n^{(m)})_{n\geq 1}$.

If we have an admissible construction, then there exists ψ such that $A(z) = \psi[B^{(1)}(z), \ldots, B^{(m)}(z)]$.

Example 2.3 (Cartesian Product)**.**

$$\mathcal{A} = \mathcal{B} \times \mathcal{C} \iff A = \{\alpha = (\beta, \gamma) | \beta \in \mathcal{B}, \gamma \in \mathcal{C}\}.$$

$$|\alpha|_{\mathcal{A}} = |\beta|_{\mathcal{B}} + |\gamma|_{\mathcal{C}}, \qquad A_n = \sum_{k=0}^{n} B_k C_{n-k}$$

$$\Rightarrow A(z) = B(z)C(z).$$

Example 2.4 (Disjoint Union of Sets)**.**

$$\mathcal{A} = \mathcal{B} \cup \mathcal{C}, \qquad \mathcal{B} \cap \mathcal{C} = \emptyset.$$

The size of $\omega \in \mathcal{A}$ is just its size as an element of $\mathcal{B}$ or $\mathcal{C}$.

$$A_n = B_n + C_n \qquad \Rightarrow \qquad A(z) = B(z) + C(z).$$

Remark 2.5. Even if

Definition 2.6. A constructible class is a class that can be defined from "primal" elements by means of admissible constructions.

Example 2.7. The *neutral class* $\mathcal{E}$ has one object of size 0.

$$\mathcal{A} \cong \mathcal{E} \times \mathcal{A} \cong \mathcal{A} \times \mathcal{E} \Rightarrow A(z) = 1.$$

The *atomic class* $\mathcal{Z}$ has one object of size 1. Its generating function is $Z(z) = z$.

Basic admissibility:

The combinatorial union, cartesian product, SEQ, PSET, MSET, CYC are all admissible with the operators on ordinary generating functions:

(1) $\mathcal{A} = \mathcal{B} + \mathcal{C}$. If the sets are not disjoint, we simply color them by taking the Cartesian product each with a particular color. Then, perform the usual disjoint union. The ordinary generating functions satisfy: $A(z) = B(z) + C(z)$.

(2) $\mathcal{A} = \mathcal{B} \times \mathcal{C}$. As we saw before, $A(z) = B(z)C(z)$.

(3) $SEQ(\mathcal{B}) = \{\epsilon\} + \mathcal{B} + \mathcal{B} \times \mathcal{B} + \cdots$.

$$\mathcal{A} = SEQ(\mathcal{B}) = \{\alpha = (\beta_1, \ldots, \beta_l) | l \geq 0\}.$$

The size function has $|\alpha|_A = \sum_{i=1}^{l} |\beta_i|_{\mathcal{B}}$ is admissible iff $\mathcal{B}$ has no objects of size 0.

$$SEQ(\mathcal{Z}) \setminus \epsilon = \bullet, \bullet\bullet, \bullet\bullet\bullet, \cdots,$$

simply the unary representation of integers ≥ 1. As for the ordinary generating functions,

$$SEQ(B) : A(z) = 1 + B(z) + B(z)^2 + \cdots = \frac{1}{1 - B(z)}.$$

(4) $\mathcal{A} = PSET(\mathcal{B}) :=$ set of all finite subsets of $\mathcal{B}$.

$$\alpha = \{\{\beta_1, \ldots, \beta_l\} | \beta_i \in \mathcal{B}\}.$$

The size is given by

$$|\alpha|_{\mathcal{A}} = \sum_{i=1}^{l} |\beta_i|.$$

$$\mathcal{A} = PSET(\mathcal{B}) := \prod_{\beta \in \mathcal{B}} (\{\epsilon\} + \{\beta\}).$$

$$A(z) = \prod_{\beta \in \mathcal{B}} (1 + z^{|\beta|}) = \prod_{n \geq 1} (1 + z^n)^{B_n}.$$

$$\Rightarrow \exp \log A(z) = \exp(\sum_{n \geq 1} B_n \log(1 + z^n)) = \exp(\sum_{k \geq 1} \frac{(-1)^{k-1}}{k} B(z^k)).$$

(5) $MSET(\mathcal{B})$:= all finite multi sets of $\mathcal{B}$. $\mathcal{B}$ is finite.

$$\mathcal{A} = MSET(\mathcal{B}) \cong \prod_{\beta \in \mathcal{B}} SEQ(\{\beta\}).$$

$$A(z) = \prod_{\beta \in \mathcal{B}} \frac{1}{1 - z^{|\beta|}}.$$

(6) Cycle construction $CYC(\mathcal{B})$:= sequences taken up to circular shift.

$$CYC(\mathcal{B}) := SEQ(\mathcal{B})/S \qquad \rightarrow \text{ circular shift.}$$

Theorem 2.8 (Symbolic Method, Unlabelled)**.** *The generating function of a "constructible class" is a completion of a system of functional equations whose terms are built from:*

$$1, z, +, \times, Q, Exp, \overline{Exp}, \log,$$

where the non-obvious operations are defined by:

$$Q[f] = \frac{1}{1 - f} \qquad (SEQ).$$

$$Exp[f] = \exp(\sum_{k \geq 1} \frac{f(z^k)}{k}) \qquad (MSET).$$

$$\overline{Exp}[f] = \exp(\sum_{k \geq 1} \frac{(-1)^{k-1} f(z^k)}{k}) \qquad (PSET).$$

$$\log[f] = \sum_{k \geq 1} \frac{\varphi(k)}{k} \log \frac{1}{1 - f(z^k)}.$$

Iterative Combinatorial Classes:

Example 2.9 (Binary Words)**.** Let $\mathcal{W}$ be the set of binary words, i.e. words on the alphabet $\{a, b\}$.

$$\mathcal{W} = SEQ(z + z) \Rightarrow W(z) = \frac{1}{1 - 2z}.$$

Example 2.10 (General Plane Trees)**.** A plane tree has a root, and its children could be considered an ordered set of roots for new trees. Therefore, we solve recursively.

$$\mathcal{G} = \mathcal{Z} SEQ(\mathcal{G}) \Rightarrow G(z) = z \frac{1}{1 - G(z)} \Rightarrow \frac{1}{2}(1 - \sqrt{1 - 4z}).$$

Example 2.11 (Polygon Triangulations)**.** An $(m+2)$-gon can be tiled by m triangles.

$$\mathcal{T} = \{\epsilon\} + (\mathcal{T}, \Delta, \mathcal{T}).$$

$$\Rightarrow T(z) = 1 + zT^2(z) \Rightarrow T(z) = \frac{1 - \sqrt{1 - 4z}}{2z}.$$

$$\Rightarrow T_n = C_n.$$

3. Thursday, June 14, 2012

Symbolic Method for Labeled Structures.

Recall that the number of trees on $n+1$ internal vertices satisfies:

$$C_n \sim \frac{4^n}{\sqrt{\pi n^3}},$$

where the 4 is related to the singularity at $1/4$.

Definition 3.1. A *labelled class* has labelled objects.

(1) Weakly-labelled objects: labeled graph whose vertices $\subset I$.
(2) Well-labelled objects of size $[n]$ if the set of labels is $[1 \dots n]$.

A labelled class is a combinatorial class of well-labelled objects.

Example 3.2 (Labelled graphs)**.** The graph on four vertices $1-2-3-4$ is equivalent to $4-3-2-1$. Therefore, we have 12 labelled graphs on four vertices.

Definition 3.3. The exponential generating function (E.G.F.) of a class $\mathcal{A}$ is given by:

$$A(z) := \sum_{n \geq 0} A_n \frac{z^n}{n!} = \sum_{\alpha \in A} \frac{z^{|\alpha|}}{|\alpha|!}.$$

The neutral class $\mathcal{E}$ and atomic class $\mathcal{Z}$ are the same as in the unlabeled universe.

Three Basic Classes of Labels

Example 3.4. The class of Permutations $\mathcal{P}$, is given by:

$$\mathcal{P} = \{\epsilon, (1), \binom{12}{21}, \dots\}.$$

Clearly, $P_n = n!$, so the exponential generating function is

$$P(z) = \sum_{n \geq 0} P_n \frac{z^n}{n!} = \sum_{n \geq 0} z^n = \frac{1}{1-z}.$$

Example 3.5. The class of Urns $\mathcal{U}$, is given by:

$$\mathcal{U} = \{\epsilon, (1), (12), (123), \dots\}.$$

$U_n = 1$, so the exponential generating function is

$$U(z) = \sum_{n \geq 0} U_n \frac{z^n}{n!} = \sum_{n \geq 0} \frac{z^n}{n!} = e^z.$$

Example 3.6. The class of Circular graphs $\mathcal{C}$, is given by:

$$\mathcal{C} = \{\epsilon, (1), (12), \binom{123}{132}, \dots\}.$$

$C_n = (n-1)!$, so the exponential generating function is

$$C(z) = \sum_{n \geq 0} C_n \frac{z^n}{n!} = \sum_{n \geq 0} \frac{z^n}{n} = \log \frac{1}{1-z}.$$

Admissible Labeled Constructions

Remark 3.7. Product of $A(z)$ and $B(z)$:

$$A(z)B(z) = C(Z) \Rightarrow C(z) = \sum_{n\geq 0} a_n \frac{z^n}{n!} \cdot \sum_{n\geq 0} b_n \frac{z^n}{n!}.$$

$$\Rightarrow C(z) = \sum_{n\geq 0} \left(\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \right) \frac{z^n}{n!}.$$

Remark 3.8. Product of multiple functions:

$$A(z) = B^{(1)}(z) \cdots B^{(r)}(z) \Rightarrow A(z) = \sum_{n\geq 0} \left(\sum_{k_1+\ldots+k_r=n} \binom{n}{k_1, k_2, \ldots, k_r} b_{k_1}^{(1)} \cdots b_{k_r}^{(r)} \right) \frac{z^n}{n!}.$$

Definition 3.9. $\beta * \gamma := \{(\beta', \gamma') \,|\, \text{the well-labeled objects } \rho(\beta') = \beta, \rho(\gamma') = \gamma\}$.

Example 3.10. Product of a labeled triangle graph and a labeled edge graph.

Definition 3.11. The labelled product $\mathcal{B} * \mathcal{C}$ is obtained by forming ordered pairs from $\mathcal{B} \times \mathcal{C}$ and performing all possible order-preserving relabeling.

$$\mathcal{A} = \mathcal{B} * \mathcal{C} = \bigcup_{\beta\in\mathcal{B},\gamma\in\mathcal{C}} (\beta * \gamma).$$

$$\Rightarrow A_n = \sum_{|\beta|+|\gamma|=n} \binom{|\beta| + |\gamma|}{|\beta|} \Rightarrow A(z) = B(z)C(z).$$

The list of admissible labeled constructions:

(1) Labeled Product.
(2) The set of k-sequences (k-SEQ), and the set of all sequences (SEQ).
(3) The set of k-Sets (k-Set), and the set of all sets (SET).
(4) The set of k-Cycles (k-CYC), and the set of all cycles (CYC).
(5) Combinatorial Sum.

The corresponding generating functions are:

(1)
$$A(z) \cdot B(z)$$

(2)
$$\frac{1}{1 - B(z)}$$

(3)
$$e^{B(z)}$$

(4)
$$\log \frac{1}{1 - B(z)}$$

(5)
$$A(z) + B(z)$$

Example 3.12 (Surjective functions)**.** Consider $\mathcal{R}$ as the set of surjections $f : A \to B$, where $|B| = r$. Let $R_n^{(r)}$ = all surjections $f : [n] \to [r]$. We can write this as the sequence $(f(1), f(2), \dots, f(n))$ all of which are elements of $[r]$. We can also think of it as the sequence of subsets of $[n]$ that map to k. For example, given

$$\binom{123456789}{212535354} \to (\{2\}, \{1,3\}, \{5,7\}, \{9\}, \{4,6,8\}).$$

Therefore, $\mathcal{R} = \mathrm{SEQ}_r(\mathrm{SET}_{\geq 1}(Z))$. This returns a generating function:

$$R(z) = (e^z - 1)^r.$$

Example 3.13 (Set Partitions)**.** Let $S_n^{(r)}$ = the number of ways to partition $[n]$ into r blocks.

$$\mathcal{S}^r = \bigcup_n \mathcal{S}_n^{(r)} = SET_r(SET_{\geq 1}(\mathcal{Z})).$$

$$\Rightarrow S^{(r)}(z) = \frac{1}{r!}(e^z - 1)^r.$$

Example 3.14 (Words)**.** Let $X = \{a_1, \dots, a_r\}$ be an alphabet. Let $\mathcal{W}$ be the class of all words, and $\mathcal{W}_n$ be the class of all words of size n.

$$\mathcal{W} = \mathcal{U}^r = SEQ_r(\mathcal{U}) \Rightarrow W(z) = e^{rz}.$$

Example 3.15 (Non-Plane Labeled Rooted Trees)**.** Let $\mathcal{T}$ be the class of all non-plane (i.e. subtrees are not ordered) labeled rooted trees.

$$\mathcal{T} = \mathcal{Z} * SEQ(\mathcal{T}).$$

4. Friday, June 15, 2012

Example 4.1. Recall that Alternating Permutations led us to the generating function $\tan(z) : \mathbb{C} \to \mathbb{C}$. We also found that

$$\lim_{z \to \pm\pi/2} T(z) \sim \frac{8z}{\pi^2 - 4z^2}.$$

These coefficients grow similarly to $2 \cdot \left(\frac{2}{\pi}\right)^{n+1} = \frac{4}{\pi}\left(\frac{2}{\pi}\right)^n$. Why does this tell us anything about the growth of the coefficients of $\tan(z)$?

The Symbolic Method has asked us to think as follows:

$$\text{Sequence} \to \text{Formal Power Series} \to \text{Functions}$$

$$\{A_n\} \to \sum A_n z^n \to A(z).$$

Example 4.2.

$$\{1, 1, 1, \dots, \} \to \sum z^n \to \frac{1}{1-z}.$$

$$\{1, 2, 4, 8, \dots\} \to \sum 2^n z^n \to \frac{1}{1-2z}.$$

Today we will concern ourselves more with the functions on the right.

In general the sequence gives us a function that is analytic on a neighborhood of 0 in $\mathbb{C}$. The analytic structure of $A(z)$ tells us about the asymptotics of A_n.

Let f be an analytic function on a neighborhood of 0. Then f has a power series representation $f(z) = \sum f_n z^n$.

Singularities of f:

Let $f : \Omega \to \mathbb{C}$. f is analytic on Ω if at every $z_0 \in \Omega$, it has a power series representations $f(z) = \sum c_n (z - z_0)^n$ on $|z - z_0| < \delta$. Equivalently, f is differentiable at each $z_0 \in \Omega$.

A singularity of f is a point where f cannot be extended analytically.

Example 4.3.

$$\{1, 1, \ldots\} \to \sum z^n \to \frac{1}{1-z}.$$

This power series converges only on $\Omega = |z| < 1$, and the function is defined and analytic on $\mathbb{C} \setminus \{1\}$. This leads to a singularity at $z_0 = 1$.

Example 4.4. The exponential generating function for derangements:

$$f_n = [z^n] f(z), \qquad f(z) = \frac{e^{-z}}{1-z} \qquad \Rightarrow D_n = n! f_n.$$

This has a singularity at $z_0 = 1$.

Example 4.5. The exponential generating function for surjections:

$$g(z) = \frac{1}{2 - e^z}.$$

This has a singularity when $e^z = 2$, i.e. for all $k \in \mathbb{Z}$,

$$z = \log 2 + 2\pi i k$$

is a singular point.

Example 4.6. The ordinary generating function for Catalan numbers C_n:

$$h(z) = \frac{1 - \sqrt{1 - 4z}}{2}.$$

$\omega \mapsto \sqrt{\omega}$ has a singularity at $\omega_0 = 0$. Therefore, this O.G.F. has a singularity at $z_0 = \frac{1}{4}$. is a singular point.

First Principle: The exponential growth of f_n tells us the *location* of singularities of f. By location, we mean "how close is the closest singularity to zero?"

Second Principle: The sub-exponential growth of f_n tells us the *nature* of singularities of f.

More precisely, suppose $f(z) = \sum f_n z^n$ has radius of convergence R such that $0 < R < \infty$. Then:

$$f_n = \left(\frac{1}{R}\right)^n \Theta(n), \text{ where } \limsup_{n \to \infty} |\Theta(n)|^{1/n} = 1.$$

More explicitly,

$$|\Theta(n)| \geq (1 + \epsilon)^n \text{ for finitely many } n,$$
$$|\Theta(n)| \geq (1 - \epsilon)^n \text{ for infinitely many } n.$$

Fact 4.7. *The closest singularity to* 0 *of* f *is on the circle* $|z| = R$.

One can justify this by assuming the negative and applying Cauchy's Integral Formula.

Exercise 4.8. Prove: If $\Theta(n) := R^n f_n$, then $\limsup |\Theta(n)|^{1/n} = 1$. Hint: Root test.

Example 4.9.

f	$d=R$	$Exp.$
$\frac{1}{1-z}$	1	1^n
$\frac{1}{1-2z}$	$1/2$	2^n
$\frac{1-\sqrt{1-4z}}{2}$	$1/4$	4^n
$\frac{1}{2-e^z}$	$\log 2$	$\left(\frac{1}{\log 2}\right)^n$

Transfer

The second principle is called "Transfer" by Flajolet-Sedgwick. We want to know

$$f(z) \sim \sigma(z) \text{ as } z \to z_0 \overset{?}{\Rightarrow} f_n \sim \sigma_n \text{ as } n \to \infty.$$

Strategy:

(1) Find singularities of $f(z)$ closest to zero. This gives you the dominant distance R. Then,

$$f_n = \left(\frac{1}{R}\right)^n \Theta(n).$$

(2) Normalize to $R = 1$ and $z_0 = 1$.
(3) Find σ in catalog such that $f \sim \sigma$ as $z \to z_0$.
(4) Conclude $f_n \sim \sigma_n$.

Example 4.10. Suppose f is analytic on $\mathbb{C} \setminus (1, \infty)$; the closest singularity is at 1. Suppose also that

$$f \sim \frac{1}{(1-z)^\alpha} = \sigma(z),$$

for $\alpha = 1, 2, 3 \ldots$. Claim: $f_n \sim \frac{n^{\alpha-1}}{(\alpha-1)!}$. We can evaluate a contour integral

$$f_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z^{n+1}} dz.$$

where γ is a "Pacman" shape with a little circle around 1 and a big circle going out to infinity, with rays at $\pm\pi/4$, as in Figure 1.

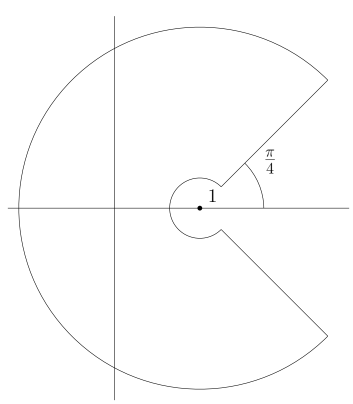

FIGURE 1. Contour for Integral.

Example 4.11. Let $\sigma = 1 - \sqrt{1-z}$, then σ_n asymptotically approaches $\frac{1}{2\sqrt{\pi n^3}}$.

Therefore, for $f = \frac{1}{2}(1 - \sqrt{1-4z})$, we have $f_n \sim 4^n \frac{1}{4\sqrt{\pi n^3}}$.

Example 4.12. Let $\sigma = \frac{1}{1-z}$, then σ_n asymptotically approaches (and always equals) 1. Let $\sigma = \frac{1}{(1-z)^2}$, then σ_n asymptotically approaches (and always equals) n.