ECCO 2012: Positive Grassmannian. Exercises Lecture 3

1.

(a) Prove the recurrence for derangement numbers D_n and the numbers of decorated permutations N_n: $D_n = nD_{n-1} + (-1)^n$ and $N_n = nN_{n-1} + 1$

(b) ($\star\star$) Can you find a bijective proof of these recurrences?

2. Positroids of rank 3 which are given by matrices $[v_1, \ldots, v_n]$ such that $v_i \neq 0$ and $v_i \not\parallel v_j$ for all i and j correspond to pictures like the one in Figure 1(a).

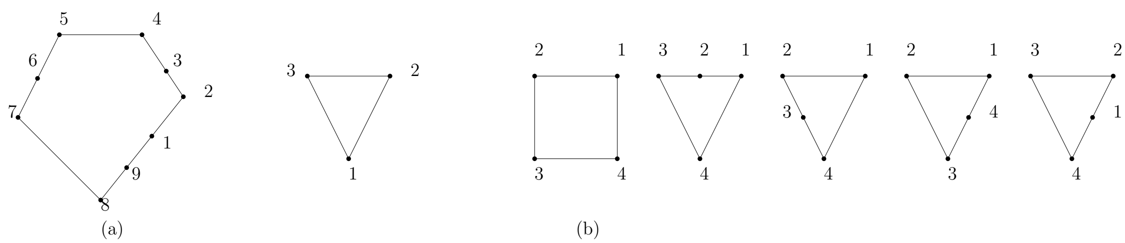

Figure 1: (a) n-gon illustrating rank 3 positroid. (b) the n-gons illustrating rank 3 positroids for $n = 3, 4$.

Let p_n be the number of such pictures with n points. For instance $p_3 = 1$ (the triangle) and $p_4 = 5$ (see Figure 1(b))

(a) Calculate p_3, p_4, p_5, p_6.

(b) Find a closed formula for p_n.

3. Let f_n be the number of hook diagrams of shape $2 \times n$.

(a) Calculate f_1, f_2, f_3.

(b) Show that $f_n = 3f_{n-1} + 2^{n-1}$, for $n \geq 1$. (Look at possible types of the last column $\begin{array}{|c|}\hline \bullet \\ \hline \bullet \\ \hline \end{array}$, $\begin{array}{|c|}\hline \circ \\ \hline \bullet \\ \hline \end{array}$, $\begin{array}{|c|}\hline \bullet \\ \hline \circ \\ \hline \end{array}$, $\begin{array}{|c|}\hline \circ \\ \hline \circ \\ \hline \end{array}$. What condition does it impose on other columns?).

(c) Let $f(x) = \sum_{n \geq 0} f_n x^n$. Deduce that $f(x) = 1 + 3xf(x) + \frac{x}{1-2x}$ and that $f(x) = \frac{(1-x)}{(1-2x)(1-3x)}$.

(d) Find a closed formula for f_n.

4. Let g_n be the number of fillings of a Young diagram $2 \times n$ with $+$'s and $-$'s with two forbidden patterns $\begin{array}{ccc} \boxed{+} & - - & \boxed{-} \\ \boxed{-} & - - & \boxed{+} \end{array}$ $\quad\begin{array}{ccc} \boxed{-} & - - & \boxed{+} \\ \boxed{+} & - - & \boxed{-} \end{array}$ (boxes may not be adjacent).

(a) Calculate g_1, g_2 and g_3.

(b) Replace columns as such by four letters following the rules: $\begin{array}{|c|}\hline + \\ \hline + \\ \hline \end{array} \mapsto A$, $\begin{array}{|c|}\hline - \\ \hline - \\ \hline \end{array} \mapsto B$, $\begin{array}{|c|}\hline + \\ \hline - \\ \hline \end{array} \mapsto C$, $\begin{array}{|c|}\hline - \\ \hline + \\ \hline \end{array} \mapsto D$. Deduce that g_n is the number of words w of length n in four letters A, B, C, D such that w cannot contain both letters C and D at the same time.

(c) Find a closed formula for g_n.

5. ($\star\star$) Prove that the number of hook diagrams of shape λ equals the number of fillings of λ with $+$ and $-$ that avoid the patterns:

$$\begin{array}{ccc} \boxed{+} & - - & \boxed{-} \\ \vdots & & \vdots \\ \boxed{-} & - - & \boxed{+} \end{array} \qquad \begin{array}{ccc} \boxed{-} & - - & \boxed{+} \\ \vdots & & \vdots \\ \boxed{+} & - - & \boxed{-} \end{array}$$