1. Recall the notation and setup of the Plücker relations: let \([n] := \{1, 2, \ldots, n\}\) and \(\binom{[n]}{k} = \{ I \mid I \subset [n], |I| = k \}\). For a \(k \times n\) matrix \(A\) and \(I = \{i_1, \ldots, i_k\} \subset [n]\), let \(\Delta_I(A) = \det(k \times k \text{ submatrix in column set } I)\). Then the Plücker relations are: for any \(i_1, \ldots, i_k, j_1, \ldots, j_k \in [n]\) and \(r = 1, \ldots, k\):
\[
\Delta_{i_1, \ldots, i_k, j_1, \ldots, j_k} = \sum \Delta_{i_1', \ldots, i_k', j_1', \ldots, j_k'} \Delta_{j_1', \ldots, j_k'}\, \tag{0.1}
\]
where we sum over all indices \(i_1, \ldots, i_k\) and \(j_1', \ldots, j_k'\) obtained from \(i_1, \ldots, i_k\) and \(j_1, \ldots, j_k\) by switching \(i_{s_1}, i_{s_2}, \ldots, i_{s_r}\) (\(s_1 < s_2 < \ldots < s_r\)) with \(j_1, j_2, \ldots, j_r\).

Prove the Plücker relation.

2. 
(a) Recall that the Fano plane is an example of a non-realizable matroid in \(\binom{[7]}{3}\) (it is illustrated in Figure 1).

Check that the Fano plane satisfies the Exchange Axiom and that it is non-realizable.

(b) Two other examples of non-realizable matroids are the Pappus matroid and the Desargues matroid (illustrated in Figures 2 3) which come from Pappus and Desargues theorems respectively. We require that the 3 points that are supposed to be collinear in Pappus/Desargues theorems are linearly independent in the corresponding Pappus/Desargues matroids.

Check that these are non-realizable matroids.

3. Let \(\lambda = (\lambda_1, \ldots, \lambda_k)\) be a Young diagram that fit inside the \(k \times n\) rectangle. Consider the subset \(S_\lambda\) of the Grassmannian \(\Gr(k, n)\) over the finite field \(\mathbb{F}_q\) that consists of elements that can
be represented by \( k \times n \) matrices \( A \) with 0s outside the shape \( \lambda \). For example, for \( n = 4 \) and \( k = 2 \), \( S_{(4,1)} \) is the subset of elements of \( \text{Gr}(2, 4) \) representable by matrices of the form \( \begin{pmatrix} \ast & \ast & \ast & \ast \\ \ast & 0 & 0 & 0 \end{pmatrix} \).

**Find** a combinatorial expression for the number of elements of \( S_{(2k, 2k-2, \ldots, 2)} \) (over \( \mathbb{F}_q \)). Show that it is a polynomial in \( q \).

4. **Recall the notation of matroid polytopes.** We denote by \( e_1, \ldots, e_n \) the coordinate vectors in \( \mathbb{R}^n \). Given \( I = \{i_1, \ldots, i_k\} \in \binom{[n]}{k} \) we denote by \( e_I \) the vector \( e_{i_1} + e_{i_2} + \cdots + e_{i_k} \). Then for any \( M \subseteq \binom{[n]}{k} \) we obtain the following convex polytope

\[
P_M = \text{conv}(e_I \mid I \in M) \subset \mathbb{R}^n,
\]

where \( \text{conv} \) means the convex hull. Note that \( P_M \subset \{x_1 + x_2 + \cdots + x_n = k\} \) so \( \dim P_M \leq n - 1 \). The polytope \( P_M \) is a **matroid polytope** if every edge of \( P_M \) is parallel to \( e_j - e_i \), i.e. edges are of the form \([e_I, e_J]\) where \( J = (I \setminus \{i\}) \cup \{j\} \).

**Prove** that \( P_M \) is a matroid polytope if and only if \( M \) satisfies the **Exchange Axiom**: For all \( I, J \in M \) and for all \( i \in I \) there exists a \( j \in J \) such that \( (I \setminus \{i\}) \cup \{j\} \in M \).