ECCO 2012: Positive Grassmannian. Exercises Lecture 1

1. *Recall the notation and setup of the* **Plücker relations***: let* $[n] := \{1, 2, \ldots, n\}$ *and* $\binom{[n]}{k} = \{I \mid I \subset [n], |I| = k\}$*. For a* $k \times n$ *matrix* A *and* $I = \{i_1, \ldots, i_k\} \subset [n]$*, let* $\Delta_I(A) = \det(k \times k$ *submatrix in column set* $I)$*. Then the Plücker relations are: for any* $i_1, \ldots, i_k, j_1, \ldots, j_k \in [n]$ *and* $r = 1, \ldots, k$*:*

$$\Delta_{i_1,\ldots,i_k,j_1,\ldots,j_k} = \sum \Delta_{i'_1,\ldots,i'_k} \Delta_{j'_1,\ldots,j'_k}, \tag{0.1}$$

where we sum over all indices $i_1, \ldots, i_k$ *and* $j'_1, \ldots, j'_k$ *obtained from* $i_1, \ldots, i_k$ *and* $j_1, \ldots, j_k$ *by switching* $i_{s_1}, i_{s_2}, \ldots, i_{s_r}$ $(s_1 < s_2 < \ldots < s_r)$ *with* $j_1, j_2, \ldots, j_r$*.*

Prove the Plücker relation.

2.

(a) *Recall that the* **Fano plane** *is an example of a non-realizable matroid in* $\binom{[7]}{3}$ *(it is illustrated in Figure 1).*

Check that the Fano plane satisfies the **Exchange Axiom** and that it is non-realizable.

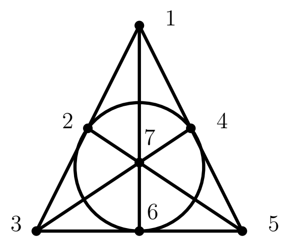

Figure 1: The Fano plane.

(b) *Two other examples of non-realizable matroids are the* **Pappus matroid** *and the* **Desargues matroid** *(illustrated in Figures 2 3) which come from Pappus and Desargues theorems respectively. We require that the 3 points that are supposed to be collinear in Pappus/Desargues theorems are linearly independent in the corresponding Pappus/Desargues matroids.*

Check that these are non-realizable matroids.

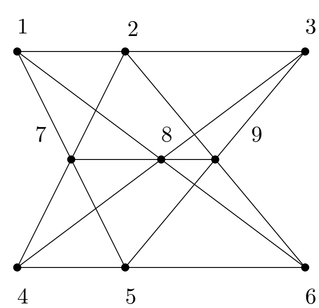

Figure 2: The Pappus matroid

3. *Let* $\lambda = (\lambda_1, \ldots, \lambda_k)$ *be a Young diagram that fit inside the* $k \times n$ *rectangle. Consider the subset* S_λ *of the Grassmannian* $\mathbf{Gr}(k, n)$ *over the finite field* $\mathbb{F}_q$ *that consists of elements that can*

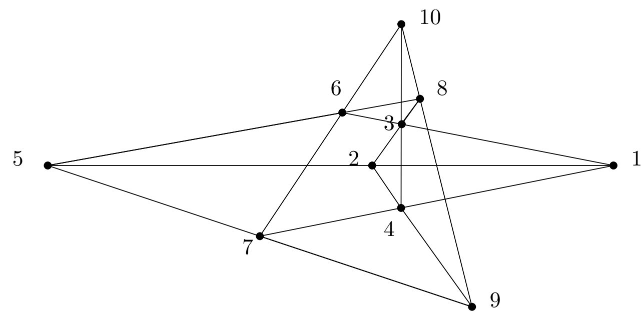

Figure 3: The Desargues matroid

be represented by $k \times n$ *matrices* A *with* 0*s outside the shape* λ*. For example, for* $n = 4$ *and* $k = 2$*,* $S_{(4,1)}$ *is the subset of elements of* $\mathbf{Gr}(2,4)$ *representable by matrices of the form* $\begin{pmatrix} * & * & * & * \\ * & 0 & 0 & 0 \end{pmatrix}$.

Find a combinatorial expression for the number of elements of $S_{(2k,2k-2,\dots,2)}$ (over $\mathbb{F}_q$). Show that it is a polynomial in q.

4. *Recall the notation of* **matroid polytopes**. *We denote by* $e_1, \dots, e_n$ *the coordinate vectors in* $\mathbb{R}^n$*. Given* $I = \{i_1, \dots, i_k\} \in \binom{[n]}{k}$ *we denote by* e_I *the vector* $e_{i_1} + e_{i_2} + \cdots + e_{i_k}$*. Then for any* $\mathcal{M} \subseteq \binom{[n]}{k}$ *we obtain the following convex polytope*

$$P_{\mathcal{M}} = \mathrm{conv}(e_I \mid I \in \mathcal{M}) \subset \mathbb{R}^n,$$

where conv *means the convex hull. Note that* $P_{\mathcal{M}} \subset \{x_1 + x_2 + \cdots + x_n = k\}$ *so* $\dim P_{\mathcal{M}} \leq n - 1$*. The polytope* $P_{\mathcal{M}}$ *is a* **matroid polytope** *if every edge of* $P_{\mathcal{M}}$ *is parallel to* $e_j - e_i$*, i.e. edges are of the form* $[e_I, e_J]$ *where* $J = (I \backslash \{i\}) \cup \{j\}$*.*

Prove that $P_{\mathcal{M}}$ is a matroid polytope if and only if $\mathcal{M}$ satisfies the **Exchange Axiom:** For all $I, J \in \mathcal{M}$ and for all $i \in I$ there exists a $j \in J$ such that $(I \backslash \{i\}) \cup \{j\} \in \mathcal{M}$.