Virtual and Γ Species

David Snyder

Before we start, I want to make it clear that this paper is mainly excerpts from sources which are referenced at the end.

We will start off reviewing the basics of the Theory of Combinatorial Species.

Let B denote the category of finite sets where the morphisms are bijections.

Definition : A combinatorial species is a functor

$$F : B \to B$$

With this, we can associate a finite set, U, with a finite set $F[V]$, which is the set of F-structures of F on A. Since F is a functor, it maintains the structures on the bijections as well. Let $\sigma : U \to V$ be a bijection. Then, $F[\sigma] : F[U] \to F[V]$ is also a bijection. Let $\tau : V \to U$ be a bijection. Since F is a functor, $F[\sigma \circ \tau] : F[\sigma] \circ F[\tau]$ is also a bijection $F[U] \to F[V]$. Let 1_U denote the identity map on U. Then $F[1_U] = 1_{F[U]}$ Some basic species are :

E which is defined by $E[u] = \{U\}$

X defined by $X[U] = U$, of $|U| = 1$, empty otherwise.

G of graphs

S of permutations

L of linear orderings

Let $a \in F[U]$ and $b \in F[V]$. Let $\sigma : U \to V$ be a bijection. σ is said to be an isomorphism of a to b if

$$b = \sigma(a) = F[\sigma](a)$$

For example we can consider the permutations of 4 letters. Let $U = V = \{1, 2, 3, 4\}$. Let $F[U]$ be the permutations on our set U. Take $\sigma : U \to U$ where $1 \mapsto 1$ $2 \mapsto 3$ $3 \mapsto 4$ and $4 \mapsto 2$. Let $a = (13)(24)$ $b = (14)(32)$. Then, we see that σ is an isomorphism.

Each species has a associated power series that helps describe the species.

- $F(X) : \sum_{i=0}^{\infty} f_n \frac{x^n}{n!}$ where f_n is the number of F structures on the set of n elements. This counts the number of labeled structures. We call this the generating series
- $F(\tilde{X}) : \sum_{n\geq 0}^{\infty} f \frac{x^n}{n!}$ Here, f is the number of unlabeled structures. We call this the type-generating series
- $Z_F(x_1, x_2, ...) = \sum_{n\geq 0} \frac{1}{n!}(\sum_{\sigma \in S_n} fixF[\sigma] x_1^{\sigma_1} x_2^{\sigma_2} ...)$ We call this the cycle index series.

Some examples are:
$L(x) = \frac{1}{1-x}$, $\tilde{L}(x) = \frac{1}{1-x}$
$E(x) = e^x$,$\tilde{E}(x) = \frac{1}{1-x}$
$X(x) = x$, $\tilde{1}(x) = x$
$G(x) = \sum_{n\geq 0} 2^{\binom{n}{2}} x^n/n!$
$S(x) = \frac{1}{1-x}$, $\tilde{S}(x) = \pi_{k=1}^{\infty} \frac{1}{1-x^k}$

When talking about equivalence of species, we are typically interested in an isomorphism between species. Notice that L and E have the same type generating series. They are not isomorphic as species. For two species to be isomorphic we require there to exist a family of bijections $\alpha_u : F[U] \to G[U]$ where $\forall \sigma$, σ is bijection

$$\begin{array}{ccc} F[U] & \xrightarrow{\alpha_u} & G[U] \\ {\scriptstyle F[\sigma]}\downarrow & & \downarrow{\scriptstyle G[\sigma]} \\ F[V] & \xrightarrow{\alpha_v} & G[V] \end{array}$$

such that the diagram above commutes.

This isomorphism will imply that all the associated series will be equal to the corresponding associated series.

In class, we have gone over various operations on species. Namely: addition, multiplication,composition,and differentiation. These operations can be extremely useful. For example, it may be more natural to view a species as a sum of two species that are better understood. However, there are operations we have not covered. These operations are a bit more complicated. Suppose you had a species A and subspecies A^x, A^y and that $A = A^x + A^y$. It would be nice if we could say $A - A^x = A^y$. It turns out this is possible, but it is necessary that we deal with subspecies. The construction of the virtual species mimics to the construction of the integers from the natural numbers. We define $Virt = (Spe \times Spe)/\sim$.

where $\sim$ is the equivalence relation defined by

$$(F,G) \sim (H,K)$$

iff $F + K \cong G + H$. To verify that $\sim$ is an equivalence relation, we will prove the additive cancellation property of species. This is exercise 3 and 4 in Combinatorial Species and Tree-like Structures. So, we will prove that $F + G = G + H \implies F = H$.

Proof. We have that $F + H = G + H$. Therefore, there exists an isomorphism, $\alpha : F + H \to G + H$. We will construct an isomorphism $\beta : F \to G$. We consider a family of bijections, $\beta_u(s) = (\alpha_u)^k(s)$, where k is the smallest integer ≥ 0 such that $(\alpha_u)^k(s) \in G[U]$. Fix U, and let $s \in F[u]$. We know that $\alpha_u(s) \in G[U] \cup H[U]$. If $\alpha_u(s) \in G[U]$, we stop. If $a_u(s) \in H[U]$, we can apply the bijection again. We proceed iteratively. Certainly the process must end as α_u is a bijection and we are dealing with finite sets. Therefore, β_u is a well-defined bijection. Now we leverage the fact that α was an isomorphism. So that $\forall s \in F[U]$, the diagram :

$$\begin{array}{ccc} F[U] & \xrightarrow{\beta_u} & G[U] \\ {\scriptstyle F[\sigma]}\downarrow & & \downarrow{\scriptstyle G[\sigma]} \\ F[V] & \xrightarrow{\beta_v} & G[V] \end{array}$$

commutes. The argument holds for all finite sets U so that β is an isomorphism. We conclude the cancellation property holds. $\square$

We typically write $[(F,G)]_\sim$ as $F - G$. Now, we will see that $\sim$ is an equivalence relation. The reflexive and symmetric properties are easily verified. Let $(F_1, G_1) \sim (F_2, G_2)$ and $(F_2, G_2) \sim (F_3, G_3)$. Then, we have that $F_1 + G_2 = G_1 + F_2$ and $F_2 + G_3 = G_2 + F_3$. This implies that $F_1 + G_2 + F_3 = G_1 + F_2 + F_3$ substituting, we have $F_1 + F_2 + G_3 = G_1 + F_2 + G_3$. By the cancellation property, we have that $F_1 + G_3 = G_1 + F_3$ so that $(F_1, G_1) \sim (F_3, G_3)$. Thus, $\sim$ is an equivalence relation.

The associated series for Virtual Species turns out to be as expected.
$(F - G)(x) = F(x) - G(x)$
$(F - G)(\tilde{x}) = F(\tilde{x}) - G(\tilde{x})$
$Z_{F-G}(x_1, x_2, ...) = Z_F(x_1, x_2, ...) - Z_G(x_1, x_2, ...)$

To see the usefulness of Virtual Species, we consider two examples. Let G denote the species of graph. Let G^c and G^d represent connected and disconnected graphs, respectively. We know that $G = G^c + G^d$. Since we are in a setting where G^c and G^d are subspecies, we can deduce $G - G^c = G^d$, which is to be expected.

A more interesting example: We consider the species E. Now there is no classical species that is the multiplicative inverse of E. However, we know that $E = \sum_{i=0}^{\infty} E_i$. So decomposing we have that $E = 1 + \sum_{i=1}^{\infty} E_i = 1 + E^+$. Therefore,

$$E^{-1} = (1 + E^+)^{-1} = \sum_{k=0}^{\infty} (-1 * E^+)^k$$

We attain the following generating series :

$$E^{-1}(x) = e^{-x}$$

$$E^{-1}(\tilde{x}) = 1 - x$$

which agrees with one's intuition.

Now we will move into Γ species. In this setting, we will let a group, Γ act on unlabeled structures of a species. More formally put, $\forall \gamma \in \Gamma$, and for every bijection σ

$$\begin{array}{ccccc}
X & \xrightarrow{F} & F[X] & \xrightarrow{\gamma_X} & F[X] \\
\downarrow \sigma & & \downarrow F(\sigma) & & \downarrow \sigma \\
Y & \xrightarrow{F} & F[Y] & \xrightarrow{\gamma_Y} & F[Y]
\end{array}$$

the diagram above commutes.

We define the Γ - cycle index for a given element $\gamma \in \Gamma$:

$$Z_F^{\Gamma}(\gamma) = \sum_{n \geq 0} \frac{1}{n!} \sum_{\sigma \in \ S_n} fix(\gamma * F[\sigma]) x_\sigma$$

note, x_σ is just the product $x_1^{\sigma_1} x_2^{\sigma_2} ... x_n^{\sigma_n}$. The coefficients of the above series count the number of fixed points of the combined action of a permutation and a group element. This is different than what we had before as we did not have a group action. Therefore, we can uncover the normal cycle index series by letting $\Gamma = \{e\}$, the trivial group. Definitions:

$$Z^{\Gamma}_{F+G}(\gamma) = Z^{\Gamma}_{F}(\gamma) + Z^{\Gamma}_{G}(\gamma)$$

$$Z^{\Gamma}_{F*G}(\gamma) = Z^{\Gamma}_{F}(\gamma) * Z^{\Gamma}_{G}(\gamma)$$

We can also define composition of two Γ species F and G to be :

$$(F \circ G)[A] = \prod_{\pi \in P(A)} (F[\pi] \times \prod_{B \in \pi} G[B])$$

Now we will venture into an example of Γ species. Let G be a bicolored graph (each vertex is assigned a color black, or white) and that each edge connects vertices of different colors. So, it is natural to let S_2 act on the species BC(bicolored graphs), where the transposition reverses the coloring of the vertices. We will compute the cycle index series for our Γ Species. So, let $\gamma = (e)$, the identity permutation. $\forall n > 0$and each permutation $\pi \in S_n$, we must count bicolored graphs on $[n]$ for which π is a color-preserving automorphism(fixed point). We will disregard empty graphs and define $BC[\emptyset] = \emptyset$. Let λ denote the cycle type of a permutation. In other words, λ is a multiset $\{1^{k_1}, 2^{k_2}, ..., n^{k_n}\}$ where $\sum_{i=1}^{n} k_i = n$. We will count bicolored graphs for which a chosen permutation π of cycle type λ is a color-preserving automorphism. For every cycle of a permutation, there is a corresponding subset of strictly white or strictly black vertices of (exclusive or). To create a bicolored graph, we draw bicolored edges into a specific vertex set. Since π is an automorphism of the graph, we will also draw all other edges in the orbit. Going forward, we will count the edge orbits for a fixed coloring, then we will consider permutations of such colorings. Since these cycle types correspond to vertex subsets, an edge connecting two cycles of lengths m and n have a corresponding orbit length of of $lcm(m, n)$. Therefore, the total number of orbits between the two cycles is $mn/lcm(m, n) = gcd(m, n)$. So, to count the number of orbits for a fixed coloring, we take the sum over the multisets of all cycle lengths m and n where m corresponds to white cycles and n corresponds to black cycles in π. We can construct any possible bicolored graph that is fixed by π by picking a subset of cycles. Thus, there are $\prod 2^{gcd(m,n)}$ graphs for a fixed coloring. There may be colorings of a graph that can allow for a permutation of a specified cycle type. So, we will split our multi-set λ into a union of white cycles and black cycles. $\lambda = w \cup b$. Therefore the number of graphs fixed by a permutation of cycle type λ , $fix(w, b) = \prod_{i \in w, j \in b} 2^{gcd(i,j)}$ This product is not taking into account the fact the the cycles are distinguishable(by color). To adjust for this, we decompose λ into i- cycles, λ_i. So, for each i cycle there are $\lambda_i C w_i = \frac{\lambda_i}{w_i! b_i!}$. Since this is done for each i cycle, we multiply to attain the total number of decompositions: $\prod_i \frac{\lambda_i}{w_i! b_i!} = z_\lambda / z_w z_b$. Therefore, for a permutation of a type λ,

$$fix(\lambda) = \frac{z_\lambda}{z_w z_b} \prod_{i,j} 2^{gcd(w_i, b_j)}$$

Note: $z_\lambda = \prod_k k^{\lambda_k} \lambda_k!$
So, in accordance to the cycle -index formula, we obtain

$$Z_{BC}^{S_2}(e) = \sum_{n>0} \sum_{w \cup b} \frac{x_{w \cup b}}{z_w z_b} \prod_{i,j} 2^{gcd(w_i, b_j)}$$

We could have arrived to this using the Poly Enumeration Theorem covered in class. Both methods are similar.

If we want to count the number of colorings fixed by a permutation and a swap of vertex colors, Γ-Species will surely help.

We look at our cycle index series for the transposition τ. We consider the cycles of the set $[n]$ induced by a permutation $\pi \in S_n$. We will count the number of bicolored graphs on $[n]$ in which $\tau * \pi$ is an automorphism. Here, τ is our transposition in S_2 that reverses the colors. Our cycle of vertices must alternate colors. Therefore, our cycles are even. From here, edges can be connected within a single cycle or between two cycles. We will count the number of these edge orbits. So, consider a cycle of length 2n. There are n^2 possible edges as there are n options for a white vertex and n options for black vertex. When n is odd, opposite vertices have the opposite color. This allows us to have an edge of length n, which corresponds to exactly one orbit of length n. If the edge is not of length n, then the orbit length is $2n$(you will hit every vertex). Since there are n^2 possible edges, our orbit length is $2n$, and there are n cases corresponding to orbits of length n, the number of these orbits is $\frac{n^2-n}{2n}$.In total, we have $\frac{n+1}{2}$ orbits when n is odd. If n is even, every orbit will be of length $2n$. So, the number of orbits is just $\frac{n}{2}$. Now, we will count the number of edges when connecting two cycles of lengths $2m$ and $2n$. The number of possible edges is $2mn$(m white n

black or m black n white). Each has orbit length $lcm(2m, 2n) = 2lcm(m, n)$. The total number of orbits is $\frac{2mn}{lcm(m,n)} = gcd(m, n)$. So, for a fixed coloring of a permutation of cycle type 2λ(corresponds to two copies of a partition λ), the number of orbits is $\sum_i \lceil \frac{\lambda_i}{2} \rceil + \sum_{i<j} gcd(\lambda_i, \lambda_j)$. Here, we use the ceiling function so we do not have separate cases of odd and even. Therefore, the total number of possible graphs for a fixed coloring is

$$\prod_i 2^{\lceil \frac{\lambda_i}{2} \rceil} \prod_{i>j} 2^{gcd(\lambda_i, \lambda_j)}$$

Given a partition 2λ with $l(\lambda)$ cycles, there are $2^{l(\lambda)}$ colorings where each cycle alternates colors. So, the number of total graphs for all permutations π of cycle type 2λ is

$$2^{l(\lambda)} \prod_i 2^{\lceil \frac{\lambda_i}{2} \rceil} \prod_{i>j} 2^{gcd(\lambda_i, \lambda_j)}$$

we have :

$$Z_{BC}^{S_2}(\tau) = \sum_{n>0} \sum_{\lambda} 2^{l(\lambda)} \frac{x_{2\lambda}}{z_{2\lambda}} \prod_i 2^{\lceil \frac{\lambda_i}{2} \rceil} \prod_{i>j} 2^{gcd(\lambda_i, \lambda_j)}$$

We end on a note about Gamma Species. While the examples, are long and technical,Gamma species comes to the rescue when it comes to graphs. The Theory of Gamma Species gave solutions to unsolved problems in graph enumeration.

References

[1] Bergeron, F., Labelle, G., Leroux, P., Readdy, M. *Combinatorial species and tree-like structures.*. Cambridge: Cambridge University Press. 1998.

[2] Gainer, Andrew Γ-*Species,Quotients, and Graph Enumeration* PhD diss., Brandeis University. 2012.

[3] Hardt, Andy, McNeely, Pete, Phan, Tung, M., J. *Combinatorial species and graph enumeration* .2013. Retrieved from https://arxiv.org/abs/1312.0542