Illuminating the Dovetail's Lair

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MAD 6206 - Enumerative Combinatorics - Spring 2020

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Outline

1. Preliminaries
   - Introduction
   - Permutation Statistics
2. An invariant under the riffle shuffle
   - A magic trick
   - Explanation
3. *a*-shuffles
   - Generalization
   - Alternative descriptions
   - Results

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Preliminaries in Playing Cards

Some facts

1. A "standard" playing card deck consists of 52 cards, consisting of 4 suits each with 13 ranks
   1. A "Set" deck consists of 81 cards
   2. A "Euchre" deck is a "stripped" deck, consisting of 24 cards
2. To ensure a game involving playing cards has an element of chance, the cards are randomized before playing each round using a process called "shuffling"

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Riffle shuffle

What is the riffle shuffle?

1. Considered the "most popular" type of shuffle
2. A deck is split into two piles of approximately equal size
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Gilbert-Shannon-Reeds model

A deck of n cards is split into two piles, where the chance that there are k cards in a pile is given by the binomial distribution

$$P(X = k) = \frac{\binom{n}{k}}{2^n} \quad \text{for } 0 \leq k \leq n$$

The two piles are interleaved together such that the probability that a card drops from the left or right pile is proportional to the number of cards in the pile. More precisely, if there are A cards in the left pile and B cards in the right piles, the chance that the next card will drop from the left pile is $A/(A + B)$ (and $B/(A + B)$ for the right pile).

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Definitions

Descents/Ascents

General note: All permutations will be given in vector notation unless otherwise specified.

Given a permutation $\pi \in S_n$, a **descent** of π is a position i where $\pi(i) > \pi(i+1)$.
Similarly, an **ascent** of π is a position i where $\pi(i) < \pi(i+1)$.

Define the **descent set** of π as

$$\text{des}(\pi) = \{i \in [n] : \pi(i) > \pi(i+1)\}$$

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This can be equivalently defined replacing $\mathrm{des}(\pi)$ with $\mathrm{asc}(\pi)$, since $A(n,k) = A(n, n-k-1)$.

Example

$A(3,1) = 4$, since there are 4 permutations in S_3 with 1 descent:

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Rising sequences do not intersect, so π can be expressed as the disjoint union of its rising sequences.

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Properties

Property 1

The Eulerian numbers satisfy the recurrence

$$A(n,k) = (k+1)A(n-1,k) + (n-k)A(n-1,k-1)$$

Proof

Let $\pi \in S_n$ such that $\mathrm{des}(\pi) = k$, and let $i \in [n]$ be such that $\pi(i) = n$. Removing $\pi(i)$ gives a new permutation with either k or $k-1$ descents, so we can consider π as a permutation of $[n-1]$ with k or $k-1$ descents with an insertion of n.

If $\pi' \in S_{n-1}$ such that π' has k descents, then in order to preserve the number of descents, n must be inserted at one of the k descents, or at the end of the permutation. Therefore π' produces $k+1$ permutations in S_n with k descents.

On the other hand, if $\pi' \in S_{n-1}$ such that π' has $k-1$ descents, in order to increase the number of descents by 1, n must be inserted either at the beginning, or at an ascent. There are $n-k-1$ ascents, so π' produces $n-k$ permutations in S_n with k descents.

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For $a \geq n$,

$$a^n = \sum_{k=0}^{n} A(n,k) \binom{a+k}{n}$$

Property 2

Let $\pi \in S_n$. Then,

$$\#\{\text{rising sequences of } \pi\} = \#\,\mathrm{des}(\pi^{-1}) + 1$$

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Example

Let

$$\pi = 6124753 \quad \Rightarrow \quad \pi^{-1} = 2374615$$

Then

$$\begin{aligned}\#\{\text{rising sequences of } \pi\} &= 3 \\ &= \operatorname{des}(\pi^{-1}) + 1\end{aligned}$$

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A magic trick

Premo by Jordan

Imagine the following scenario:
You are given a brand new deck of cards, and you are instructed to

- Cut the deck
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- Select the top card (without looking) and insert it into the deck
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After performing these steps, you are allowed to analyze the deck. Can you determine which card was inserted into the deck?

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Yes! (Well maybe, with 84% probability).

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Explanation

The key is rising sequences

Suppose that the deck contains n cards, and it starts in its original orientation; there is one rising sequence. Then performing a riffle shuffle with k cards in one pile and $n - k$ cards in the other pile will interleave the cards such that the cards $1, 2, \ldots, k$ will form a rising sequence and the cards $k + 1, k + 2, \ldots, n$ will form a rising sequence. In this manner, subsequent riffle shuffles will approximately double the number of rising sequences, until a certain number (determined by n).

In the Premo setup, $n = 52$, and performing 3 riffle shuffles will generally yield $2^3 = 8$ rising sequences.

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Cuts

However, in the Premo setup, cuts are performed alongside the riffle shuffle. In practice, a cut will change the number of rising sequences in the shuffle.

This issue can be resolved by thinking of the shuffled deck as a loop. Then cutting the deck corresponds to rotating the loop, while riffle shuffling the deck corresponds to doubling the loop onto itself.

Definition

Define the **winding number** of a deck to be the number of times required to cycle through the deck by successive ranks.

The deck in its original orientation has winding number 1, and each riffle shuffle (up to a certain number) will increase the winding number by 1.

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Winding numbers

Let $\pi(i)$ be the position of card i, and let $d(i,j)$ be the least positive integer so $d(i,j) = \pi(j) - \pi(i) \mod n$. Then, label the card i with the count $d(i-1,i) + d(i,i+1) - 1$. If the trick worked as planned, then the target card will be the only card whose label exceeds n. This corresponds to the only card which lies on its own cycle when computing the winding number.

m	2	3	4	5	6	7	8	9	10	11	12	∞
1	997	839	288	088	042	028	023	021	020	020	019	019
2	1000	943	471	168	083	057	047	042	040	039	039	038
3	1000	965	590	238	123	085	070	063	061	059	058	058
13	1000	998	884	617	427	334	290	270	260	254	252	250
26	1000	999	975	835	688	596	548	524	513	505	503	500

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Generalizing to a higher order

a-shuffles

While the riffle shuffle as previously described is very commonplace, there exists a higher-order variant of the riffle shuffle which generalizes to *a* piles (for $a > 2$).

An *a*-**shuffle** takes a deck of cards, cuts it into *a* piles and interleaves the piles together in some order.

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The Gilbert-Shannon-Reeds model for the *a*-shuffle takes a deck of n cards, cuts it into piles of size $j_1, \ldots, j_a$ such that $j_1 + \ldots + j_a = n$, such that the probability of having piles of size $j_1, \ldots, j_a$ is given by the multinomial distribution

$$P(X_1 = j_1, \ldots, X_a = j_a) = \binom{n}{j_1, \ldots, j_a} \frac{1}{a^n} = \frac{n!}{j_1! \cdots j_a!} \frac{1}{a^n}$$

The a piles are interleaved together such that if $A_1, \ldots, A_a$ cards remaining in the a piles, the probability that the next card will drop from the i-th pile is $A_i/(A_1 + \ldots + A_a)$.

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Maximum entropy/inverse

Maximum entropy description

A **maximum entropy *a*-shuffle** is any shuffle whereby an *n* card deck is cut into *a* piles and interleaved in such a way that every possible path from the deck to the piles to the final permutation is equiprobable.

Inverse description

An **inverse *a*-shuffle** takes a deck of *n* cards, holds it face down, and turns up successive cards and places them into *a* piles uniformly and independently. Finally, the *a* piles are placed on top of each other.

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A **geometric a-shuffle** will place n points uniformly and independently in the unit interval, and label them in order $x_1 < \ldots x_2 < \ldots < x_n$. For a positive integer a, the map $x \mapsto ax \mod 1$ maps the unit interval onto itself and preserves measure. The rearrangement of the points x_i is deemed a geometric a-shuffle.

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A **sequential a-shuffle** will take a deck of n cards, cut it into piles of size $j_1, \ldots, j_a$ such that $j_1 + \ldots + j_a = n$, such that the probability of having piles of size $j_1, \ldots, j_a$ is given by the multinomial distribution

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Equivalence

Lemma 1

The four descriptions of an *a*-shuffle generate the same permutation distribution.

Proof

It suffices to show that each *a*-shuffle generates a permutation given by a multinomial distribution on *a*-piles.
It is true for the sequential *a*-shuffle by construction.
It is clear for the inverse *a*-shuffle.
For the maximum entropy *a*-shuffle, note that the number of ways to cut a deck of n cards into a piles of sizes $j_1, \ldots, j_a$ is $\binom{n}{j_1,\ldots,j_a}$, all of which are equally likely. This is in 1-1 correspondence with the possible interleavings starting from a given cut.

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Proof (cont'd)

For the geometric a-shuffle, the size of the i-th pile is determined by how many points end up in the interval $[(i-1)/a, (i/a)]$. This is known to be multinomial.

Product rule

Lemma 2

In each model, an a-shuffle followed by a b-shuffle is equivalent to an ab-shuffle.

Proof

From Lemma 1, it suffices to show that an inverse a-shuffle followed by an inverse b-shuffle is equivalent to an inverse ab-shuffle. Label each card with an ordered pair, where the first coordinate indicates which pile the card was placed during the inverse a-shuffle, and the second coordinate indicates which pile the card was placed during the inverse b-shuffle. Then forming piles with cards with the same label generates an inverse ab-shuffle.

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The probability that an a-shuffle will result in the permutation π is

$$\binom{a+n-r}{n}\frac{1}{a^n}$$

where r is the number of rising sequences in π.

Proof

Let π be a possible permutation attained from the maximum entropy a-shuffle, such that π has r rising sequences. From the above discussion, note that once the n cards are assigned to a piles, after interleaving, each rising sequence is equal to the union of some piles.

So, if the shuffled deck corresponds to the permutation π, it must be that $r-1$ cuts fall between each successive rising sequence, but the remaining $a-r$ cuts can be made arbitrarily.

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Proof cont'd

Therefore, the number of ways of assigning r rising sequences into a piles is in 1-1 correspondence with the number of ways of cutting n cards $a - r$ times. Since there are $n + 1$ places to cut a deck of n cards, the number of ways to cut the deck $a - r$ times is $\left(\!\binom{n+1}{a-r}\!\right) = \binom{a+n-r}{n}$. Since there are a^n possible a-shuffles in total, we have that the probability that a-shuffled deck corresponds to the permutation π is

$$\binom{a+n-r}{n}\frac{1}{a^n}$$

Worpitzki's Identity Proof

Proof

From Theorem 3 and the axioms of probability theory,

$$\frac{\sum_{\pi} \binom{a+n-r}{n}}{a^n} = 1$$

Since there are $A(n, r)$ permutations with r rising sequences, we can rewrite the sum as

$$\frac{\sum_{r=1}^{n} A(n, r) \binom{a+n-r}{n}}{a^n} = 1$$

Multiplying both sides gives us Worpitzki's identity:

$$\sum_{r=1}^{n} A(n, r) \binom{a+n-r}{n} = a^n$$

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