

(Problems selected from worksheets by Rob Bayer)

- (1) **Direction Field Practice.** On the back of the page, there are 4 direction fields.
- Without thinking hardly at all, which one of these is for  $y' = 1 + y$ ? Why?
  - The differential equations for the other ones are  $y' = x^2 - y^2$ ,  $y' = y \sin(2x)$ , and  $y' = 1 - xy$ . Determine which is which.
  - Using the direction fields, sketch some solution curves to  $y' = x^2 - y^2$ .

**Solution:**

- The bottom right direction field. It is only dependent on  $y$ .
- $y' = x^2 - y^2$ : Top left.  $y' = y \sin(2x)$ : top right.  $y' = 1 - xy$ : bottom left.
- Follow the arrows!

(2) **Separable Equations word problems!**

- A tank initially contains 100L of water with 1000g of salt dissolved in it. Brine containing 50g/L of salt is pumped in at a rate of 2L/min. The solution is kept thoroughly mixed and solution leaves the tank at a rate of 2L/min. Set up and solve an initial value problem whose solution would give you the grams of salt in the tank at time  $t$ .  
Hint 1: The rate of change of the amount of salt is the same as (the amount of salt coming in) – (the amount of salt leaving).  
Hint 2: The amount of salt leaving depends on how much salt is in the solution now.

**Solution:** The differential equation modeling this situation is:

$$y' = 50 \text{ g/L} \cdot (2\text{L/min}) - \frac{y}{100 \text{ L}} \cdot (2\text{L/min})$$

We ignore the units for now, and solve:

$$\frac{dy}{dt} = 100 - \frac{y}{50}$$

Using separable equations, this turns into:

$$50 \int \frac{dy}{5000 - y} = \int dt \Rightarrow -50 \ln |5000 - y| = t + C \Rightarrow y = 5000 - Ae^{-t/50}$$

where  $A$  is an arbitrary nonzero constant. Bringing the units back, we have:

$$y = \left( 5000 - Ae^{-t/50 \text{ min}} \right) \text{ grams}$$

Checking  $A = 0$ , we find that the constant function  $y = 5000$  grams is an equilibrium solution, so we let  $A$  take any value.

Now we want to find the solution with the given initial condition  $y(0) = 1000$  grams. This gives  $1000 \text{ grams} = 5000 - A \text{ grams}$ . So  $A = 4000$  and  $y = 5000 - 4000e^{-t/50 \text{ min}}$  grams.

- A certain curve in the plane has the property that every normal line (that is, a line perpendicular to the tangent line) to the curve passes through  $(2, 0)$ . Find the equation for this curve if you know it passes through  $(1, 1)$ .  
Hint: What this problem is really asking you is to find a curve where at each point  $(x, y)$ , the tangent line (which has slope  $dy/dx$ ) is perpendicular to the line from  $(2, 0)$  to  $(x, y)$  (what is the slope of this line?).

**Solution:** The line from  $(x, y)$  to  $(2, 0)$  has slope  $\frac{y}{x-2}$  (rise over run!). We want the derivative to be perpendicular to this, so we set it equal to the negative reciprocal:

$$\frac{dy}{dx} = -\frac{x-2}{y} \Rightarrow \int y dy = \int 2 - x dx \Rightarrow \frac{1}{2}y^2 = 2x - \frac{1}{2}x^2 + C.$$

Plugging in the “initial value,” i.e.  $(1, 1)$ , we obtain  $1/2 = 2 - 1/2 + C \Rightarrow C = -1$ . So the equation defining the desired curve is  $y^2 = 4x - x^2 - 2$  (multiplying the whole thing by 2 for convenience).

This can also be written as  $y^2 + x^2 - 4x + 4 = 2 \Leftrightarrow y^2 + (x-2)^2 = (\sqrt{2})^2$ : the equation for a circle of radius  $\sqrt{2}$  centered at  $(2, 0)$ .

(3) Consider the differential equation  $y' = (y-3)(y+2)^2(y+4)$ .

- Without solving for  $y$ , what are the equilibrium solutions of this differential equation?
- Sketch a graph with the equilibrium solutions, and other solutions in between. (Consider where the slope is positive or negative.)
- Use separable equations to find an expression for  $x$  in terms of  $y$ . ( $y$  can't be written simply as a function of  $x$ .)

**Solution:**

- (1) We want to plug in a constant function  $y$ , such that  $y'$  as calculated by the differential equation will be zero. This works for:

$$y = 3, y = -2, y = -4$$

(3)

$$x = \frac{1}{700} \left( \frac{70}{y+2} + 4 \log |y-3| + 21 \log |y+2| - 25 \log |y+4| \right) + C.$$