

(Problems selected from worksheets by Rob Bayer.)

- (1) Determine whether each of the following sequences are convergent or divergent. For those that are convergent, find the limit.

(a) $a_n = \frac{3n^2+1}{n^2-1}$. **Convergent.** $\lim_{n \rightarrow \infty} a_n = 3$.

(b) $a_n = \frac{(n+2)!}{(2n)^2 \cdot n!}$. **Convergent.** $\lim_{n \rightarrow \infty} a_n = \frac{1}{4}$.

- (c) $\{1, \frac{1}{2}, 1, \frac{1}{4}, 1, \frac{1}{8}, \dots\}$. **Divergent.** There are infinitely many terms equal to 1, and another infinite set approaching zero. No sequence can have two limits.

(d) $a_n = \ln(n^2-3n+1) - \ln(n^2+4)$. **Convergent.** $\lim_{n \rightarrow \infty} a_n = \ln(\lim_{n \rightarrow \infty} \frac{n^2-3n+1}{n^2+4}) = \ln 1 = 0$.
We can move the limit inside because \ln is continuous at 1.

(e) $a_n = n \tan(1/n)$. **Convergent.** Apply L'Hopital's rule to the corresponding continuous function. $\lim_{n \rightarrow \infty} a_n = 1$.

- (2) True/False. For all problems, a_n and b_n are sequences. If the answer is true, cite a theorem, or explain why. If it is false, give a counterexample, i.e. two sequences for which it is false.

(a) If a_n and b_n converge, then $a_n + b_n$ converges. **True.**

(b) If $a_n + b_n$ converges, then a_n and b_n converge. **False. Counterexample:** $a_n = \frac{1}{n}$, $b_n = \frac{-1}{n}$

(c) If a_n and b_n converge, then a_n/b_n converges. **False. Counterexample:** $a_n = 1$, $b_n = \frac{1}{n}$

(d) If a_n and b_n diverge, then $a_n + b_n$ diverges. **False. Counterexample:** $a_n = n$, $b_n = -n$

(e) If $a_n + b_n$ diverges, then a_n and b_n diverge. **False. Counterexample:** $a_n = n$, $b_n = \frac{1}{n}$
Here, only a_n diverges.

(f) If a_n and b_n diverge, then $a_n b_n$ diverges. **False. Counterexample:** $a_n = \cos n$, $b_n = \sec n$

- (3) For each of the following, give an example of a sequence with the required properties or explain why no such sequence can exist:

- (a) Bounded, Monotonic, Convergent.

$$a_n = \frac{1}{n}. \quad 0 \leq a_n \leq 1. \quad a_{n+1} < a_n. \quad \lim_{n \rightarrow \infty} a_n = 0.$$

- (b) Bounded, Monotonic, Not Convergent. **By the Monotone Sequence Theorem, any bounded monotonic sequence will converge.**

- (c) Bounded, Not Monotonic, Convergent.

$$a_n = \frac{(-1)^n}{n}. \quad -1 \leq a_n \leq 1. \quad \lim_{n \rightarrow \infty} a_n = 0.$$

- (d) Bounded, Not Monotonic, Not Convergent.

$$a_n = (-1)^n. \quad -1 \leq a_n \leq 1.$$

- (e) Not Bounded, Monotonic, Convergent. **Every convergent sequence is bounded. If a_n approaches a limit L , then for some N , all of the terms a_n , $n > N$ will be between $(L - 1)$ and $(L + 1)$. Since there are finitely many terms before that, you can take the smallest and biggest, m and M . Then, the bounds will be $\min\{m, L - 1\}$ and $\max\{M, L + 1\}$.**

- (f) Not Bounded, Monotonic, Not Convergent.

$$a_n = n. \quad a_{n+1} > a_n.$$

- (g) Not Bounded, Not Monotonic, Convergent. **By part (e), the fact that it is not bounded implies that it cannot be convergent.**
- (h) Not Bounded, Not Monotonic, Not Convergent.

$$\mathbf{a_n = (-1)^n n.}$$

(4) More Sequences! Determine convergence or divergence, and calculate the limit if convergent:

(a) $a_n = \frac{\cos^2 n + n}{2^n + 3^n}$. **Convergent.** $\lim_{n \rightarrow \infty} \mathbf{a_n = 0.}$

(b) $a_n = \frac{n^{(-1)^n}}{n + \ln n}$ **Divergent.** **When $n = 2k$, the sequence is $\frac{n}{n + \ln n}$, which approaches 1. When $n = 2k + 1$, the sequence is $\frac{1}{n^2 + n \ln n}$, which approaches zero.**

(c) $a_n = n^{\frac{\ln 2}{1 + \ln n}}$. **Convergent.** $\mathbf{a_n = \exp\left(\frac{\ln n \ln 2}{1 + \ln n}\right)}$. **Because the exponential is continuous, we can take the limit of the inside.** $\lim_{n \rightarrow \infty} \mathbf{a_n = \exp\left(\lim_{n \rightarrow \infty} \frac{\ln n \ln 2}{1 + \ln n}\right) = \exp(\ln 2) = 2.}$