

1.) a.) $\int \frac{x}{\sqrt{9-x^2}} dx$ sub in $x=3\sin\theta$ and $dx=3\cos\theta d\theta$

$$= \int \frac{9\sin\theta\cos\theta d\theta}{\sqrt{9-9\sin^2\theta}} \quad \sqrt{9-9\sin^2\theta} = \sqrt{9-9\sin^2\theta} = \sqrt{9(1-\sin^2\theta)} = \sqrt{9\cos^2\theta} = 3|\cos\theta|$$

$$= \int \frac{9\sin\theta\cos\theta d\theta}{3|\cos\theta|} \quad |\cos\theta| = \cos\theta \text{ if } 0 \leq \theta \leq \frac{\pi}{2}$$

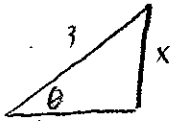
$$= \frac{9}{3} \int \frac{\sin\theta\cos\theta d\theta}{\cos\theta}$$

$$= 3 \int \sin\theta d\theta$$

$$= 3(-\cos\theta) + C$$

$$= -3\sqrt{9-x^2} + C$$

$$\boxed{-\sqrt{9-x^2} + C}$$



$$\sin\theta = \frac{x}{3} \text{ or } a^2 + x^2 = 9$$

$$a = \sqrt{9-x^2}$$

$$\cos\theta = \frac{\sqrt{9-x^2}}{3}$$

b.) $\int \frac{x}{\sqrt{9-x^2}} dx$ $u=9-x^2$ and $du=-2x dx$

$$= \frac{-1}{2} \int \frac{-2x dx}{\sqrt{9-x^2}}$$

$$= \frac{-1}{2} \int \frac{du}{u^{1/2}}$$

$$= \frac{-1}{2} \int u^{-1/2} du$$

$$= \frac{-1}{2} (2u^{1/2}) + C$$

$$= -\sqrt{u} + C$$

$$\boxed{-\sqrt{9-x^2} + C}$$

c.) The answers are the same. Yes!

2a

$$\int \frac{dx}{x^2 \sqrt{16-x^2}} \quad \begin{array}{l} x = 4 \sin \theta \\ dx = 4 \cos \theta d\theta \end{array}$$

$$= \int \frac{4 \cos \theta d\theta}{16 \sin^2 \theta \sqrt{16 - 16 \sin^2 \theta}}$$

$$= \frac{1}{4} \int \frac{\cos \theta d\theta}{\sin^2 \theta (4 \cos \theta)}$$

$$= \frac{1}{16} \int \frac{\cancel{\cos \theta} d\theta}{\sin^2 \theta \cancel{\cos \theta}}$$

$$= \frac{1}{16} \int \frac{d\theta}{\sin^2 \theta}$$

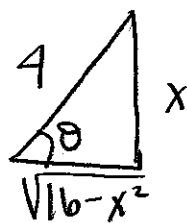
$$= \frac{1}{16} \int \csc^2 \theta d\theta$$

$$= \frac{1}{16} (-\cot \theta)$$

$$= \frac{1}{16} \left(-\cot \left(\sin^{-1} \frac{x}{4} \right) \right)$$

$$= \frac{1}{16} \left(-\frac{\sqrt{16-x^2}}{x} \right)$$

$$= \boxed{\frac{-\sqrt{16-x^2}}{16x} + C}$$



$$\theta = \sin^{-1} \frac{x}{4}$$

2) b)

$$\int \frac{\cos t}{\sqrt{1+\sin^2 t}} dt$$

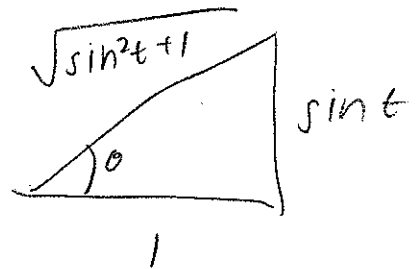
$$\begin{aligned} \sin t &= \tan \theta \\ \cos t dt &= \sec^2 \theta d\theta \end{aligned}$$

$$\int \frac{1}{\sqrt{1+\tan^2 \theta}} \cdot \sec^2 \theta d\theta \Rightarrow \int \sec \theta d\theta \Rightarrow$$

$$\ln |\sec \theta + \tan \theta| + C$$

$$\ln |\sec \theta + \sin t| + C$$

$$\boxed{\ln |\sqrt{\sin^2 t + 1} + \sin t| + C}$$



3. a) False. Let $x = 8\pi$.

$$\sin(8\pi) = 0. \quad \sin^{-1}(0) = 0 \neq 8\pi.$$

b) False. Let $x = -\frac{\pi}{4}$. $\sec(-\frac{\pi}{4}) = \sqrt{2}$.

$$\text{Then, } \sqrt{(\sqrt{2})^2 - 1} = 1.$$

On the other hand, $\tan(-\frac{\pi}{4}) = -1 \neq +1$.

c) True. On its domain, $\sin^{-1}(x)$ gives a θ that returns only this x as its sine.

$$1a. \frac{x^3 + 2x - 1}{x^2(x-4)(x+1)} = \left[\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-4} + \frac{D}{x+1} \right]$$

$$b. \frac{3x^2 - 7x}{(x-2)^2(x^2+x+1)^2} = \left[\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-2} + \frac{E}{(x-2)^2} + \frac{Fx+G}{x^2+x+1} + \frac{Hx+I}{(x^2+x+1)^2} \right]$$

$$c. \begin{array}{r} x \ 1 \\ x^2-x \overline{) x^3 + 0x^2 + 0x - 1} \\ \underline{x^3 - x^2} \\ x^2 \\ \underline{x^2 - x} \\ x - 1 \end{array}$$

$$\frac{x^3 - 1}{x^2 - x} = x + 1 + \frac{x-1}{x^2 - x} = \left[x + 1 + \frac{A}{x} + \frac{B}{x-1} \right]$$

OR

$$x + 1 + \frac{1}{x(x-1)} = \left[x + 1 + \frac{A}{x} \right]$$

Partial Fractions

$$2. a) \int \frac{dx}{x^2-4} = \int \frac{dx}{(x+2)(x-2)}$$

$$\left(\frac{1}{(x+2)(x-2)} = \frac{A}{x+2} + \frac{B}{x-2} \right) x^2-4$$

$$1 = A(x-2) + B(x+2)$$

$$(A+B)x - 2A + 2B$$

$$(A+B=0)2$$

$$2B - 2A = 1$$

$$\oplus 2A + 2B = 0$$

$$4B = 1$$

$$B = \frac{1}{4}$$

$$A = -\frac{1}{4}$$

$$= \int \left(\frac{-1}{4(x+2)} + \frac{1}{4(x-2)} \right) dx$$

$$= -\frac{1}{4} \ln|x-2| - \frac{1}{4} \ln|x+2| + C$$

$$= \frac{1}{4} \ln \left| \frac{x-2}{x+2} \right| + C$$

$$b) \int \frac{dx}{x^2-4}$$

$$x = 2 \sec \theta$$

$$dx = 2 \sec \theta \tan \theta d\theta$$

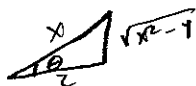
$$= \int \frac{2 \sec \theta \tan \theta}{4 \sec^2 \theta - 4} d\theta$$

$$= \int \frac{\cancel{2} \sec \theta \tan \theta}{2 \cancel{2} + \tan^2 \theta} d\theta$$

$$= \int \frac{1}{\cancel{2} \cos \theta} \cdot \frac{\cancel{\cos \theta}}{\sin \theta} \cdot \frac{1}{2} d\theta$$

$$= \int \frac{1}{2} \csc \theta d\theta$$

$$= \frac{1}{2} \ln |\csc \theta - \cot \theta| + C$$



$$= \frac{1}{2} \ln \left| \frac{x}{\sqrt{x^2-4}} - \frac{2}{\sqrt{x^2-4}} \right| + C$$

$$\frac{1}{2} \ln \left| \frac{x-2}{x^2-4} \right| + C$$

$$2. \quad c) \quad \frac{1}{4} \ln \left| \frac{x-2}{x+2} \right| + C_1 \stackrel{?}{=} \frac{1}{2} \ln \left| \frac{x-2}{\sqrt{x^2-4}} \right| + C_2.$$

$$\text{RHS} = \frac{1}{2} \ln \left| \frac{x-2}{\sqrt{(x+2)(x-2)}} \right| + C_2$$

$$= \frac{1}{2} \ln \left| \frac{\sqrt{x-2}}{\sqrt{x+2}} \right| + C_2$$

$$= \frac{1}{2} \ln \left| \left(\frac{x-2}{x+2} \right)^{1/2} \right| + C_2$$

$$= \frac{1}{4} \ln \left| \frac{x-2}{x+2} \right| + C_2 = \text{LHS} \quad \checkmark$$

QUESTION 3A - Partial Fractions

$$\int \frac{6x^3 + 7x^2 - 2x - 5}{x^4 - x^2}$$

$$= \int \frac{6x^3 + 7x^2 - 2x - 5}{x^2(x^2 - 1)} = \int \frac{6x^3 + 7x^2 - 2x - 5}{x^2(x+1)(x-1)}$$

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x+1)} + \frac{D}{(x-1)}$$

$$Ax^3 - Ax + Bx^2 - B + Cx^3 - Cx^2 + Dx^3 + Dx^2$$

$$A + C + D = 6$$

$$A = 2$$

$$B + D - C = 7 \implies$$

$$B = 5$$

$$-A = -2$$

$$D = 3$$

$$-B = -5$$

$$C = 1$$

$$\int \frac{2}{x} + \int \frac{5}{x^2} + \int \frac{1}{(x+1)} + \int \frac{3}{(x-1)}$$

$$= 2 \ln|x| - \frac{5}{x} + \ln|x+1| + 3 \ln|x-1| + C$$

$$\int \frac{3 \cdot e^{2t}}{e^{2t} - e^t - 6} \cdot dt$$

$$\begin{aligned} u &= e^t \\ du &= e^t \cdot dt \\ &= u \cdot dt \\ dt &= \frac{du}{u} \end{aligned}$$

3.(b)

$$= \int \frac{3 \cdot u^2}{u^2 - u - 6} \cdot \frac{du}{u}$$

$$\frac{3u}{u^2 - u - 6} = \frac{A}{u+2} + \frac{B}{u-3}$$

$$\begin{aligned} 3u &= A(u-3) + B(u+2) \\ &= (A+B)u + (-3A+2B) \end{aligned}$$

$$A+B=3 \quad A=3-B$$

$$-3A+2B=0 \quad -9+3B+2B=0$$

$$5B=9 \quad B=\frac{9}{5} \quad A=3-\frac{9}{5}=\frac{6}{5}$$

$$I = \int \frac{6}{5} \frac{1}{u+2} du + \int \frac{9}{5} \frac{1}{u-3} du$$

$$= \frac{6}{5} \ln|u+2| + \frac{9}{5} \ln|u-3| + C$$

$$= \frac{6}{5} \ln|e^t+2| + \frac{9}{5} \ln|e^t-3| + C$$