

Here, we will present a step-by-step approach to finding series solutions to differential equations.

- (1) Set $y = \sum c_n x^n$, and similar series notation for y' and y'' . Substitute them into the differential equation.
- (2) Simplify algebraic expressions until everything is in a power series.
- (3) Shift indices so that every series has a general term with x^n .
- (4) Pull out terms from the series so that every series begins from the same index.
- (5) Combine all of the series (this is allowed since all series are assumed to be convergent).
- (6) Figure out the equations among the coefficients that are implied by your equation. (In general, you should set the coefficient of every power of x equal to zero.)
- (7) Use these equations to find a recurrence formula for the n -th coefficient c_n . Be careful about which indices your recurrence formula apply to – this depends on where your series starts!
- (8) Write out coefficients for many values of n until you observe a pattern. Write a formula for the general term using this pattern.

Helpful Hints:

- (1) If you have a second-order differential equation, you should have exactly two arbitrary constants by the end. (If it's first-order you should have one.)
- (2) If the pattern you notice is like a factorial, but with numbers missing, write it as a factorial divided by the missing numbers.

Example:

$$c_n = \frac{c_0}{1 \cdot 2 \cdot 4 \cdot 5 \cdot 7 \cdot 8 \cdots (3n+2)} = \frac{3 \cdot 6 \cdots (3n)c_0}{(3n+2)!} = \frac{3^n n! c_0}{(3n+2)!}.$$

These can often look messy – check a few terms to make sure your formula matches the pattern.