Here, we will present a step-by-step approach to finding series solutions to differential equations.

(1) Set $y = \sum c_n x^n$, and similar series notation for y' and y''. Substitute them into the differential equation.

(2) Simplify algebraic expressions until everything is in a power series.

(3) Shift indices so that every series has a general term with x^n.

(4) Pull out terms from the series so that every series begins from the same index.

(5) Combine all of the series (this is allowed since all series are assumed to be convergent).

(6) Figure out the equations among the coefficients that are implied by your equation. (In general, you should set the coefficient of every power of x equal to zero.)

(7) Use these equations to find a recurrence formula for the n-th coefficient c_n. Be careful about which indices your recurrence formula apply to – this depends on where your series starts!

(8) Write out coefficients for many values of n until you observe a pattern. Write a formula for the general term using this pattern.

Helpful Hints:

(1) If you have a second-order differential equation, you should have exactly two arbitrary constants by the end. (If it's first-order you should have one.)

(2) If the pattern you notice is like a factorial, but with numbers missing, write it as a factorial divided by the missing numbers.

Example:

$$c_n = \frac{c_0}{1 \cdot 2 \cdot 4 \cdot 5 \cdot 7 \cdot 8 \cdots (3n+2)} = \frac{3 \cdot 6 \cdots (3n) c_0}{(3n+2)!} = \frac{3^n n! c_0}{(3n+2)!}.$$

These can often look messy – check a few terms to make sure your formula matches the pattern.