

Quiz 3

MATH 1B, SPRING 2012

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SECTION:

NAME:

Solve the following integral, showing all steps clearly:

$$\int_0^{\infty} \frac{x \arctan(x) dx}{(1+x^2)^2}$$

Be careful about identifying improper integrals, and expressing them as limits of proper integrals!

Approach 1 (Trig Sub)

First, we note that the integral is improper, so we express

$$\text{it as: } \lim_{t \rightarrow \infty} \int_0^t \frac{x \arctan(x) dx}{(1+x^2)^2} \quad \left[\begin{array}{l} \text{Let } x = \tan \theta \Rightarrow dx = \sec^2 \theta d\theta \\ 1+x^2 = 1+\tan^2 \theta = \sec^2 \theta. \end{array} \right]$$

$$= \lim_{t \rightarrow \infty} \int_0^{\arctan t} \frac{\theta \tan \theta}{\sec^4 \theta} \sec^2 \theta d\theta$$

$$= \lim_{t \rightarrow \infty} \int_0^{\arctan t} \theta \sin \theta \cos \theta d\theta = \lim_{t \rightarrow \infty} \int_0^{\arctan t} \theta \cdot \frac{1}{2} \sin 2\theta d\theta$$

$$= \lim_{t \rightarrow \infty} \left[-\frac{1}{4} \theta \cos 2\theta \Big|_0^{\arctan t} + \frac{1}{4} \int_0^{\arctan t} \cos 2\theta d\theta \right] \quad \left[\begin{array}{l} u = \frac{1}{2} \theta \quad dv = \sin 2\theta \\ du = \frac{1}{2} d\theta \quad v = -\frac{1}{2} \cos 2\theta \end{array} \right]$$

$$= \lim_{t \rightarrow \infty} \left[-\frac{1}{4} \arctan t \cdot \cos(2\arctan(t)) + \frac{1}{8} \sin(2\arctan t) \right]$$

$$= -\frac{1}{4} \cdot \frac{\pi}{2} \cdot (-1) + \frac{1}{8} \sin \pi = \frac{\pi}{8}$$

Approach 2 (IBP first)

$$\int_0^{\infty} \frac{x \arctan x}{(1+x^2)^2} dx = \lim_{t \rightarrow \infty} \int_0^t \frac{x \arctan x}{(1+x^2)^2} dx \quad (\text{the limit of a proper integral})$$

IBP:
$$\left[\begin{array}{l} u = \arctan x \quad dv = \frac{x}{(1+x^2)^2} \\ du = \frac{dx}{1+x^2} \quad v = \int \frac{x dx}{(1+x^2)^2} = \frac{1}{2} \int \frac{dw}{w^2} = -\frac{1}{2w} = -\frac{1}{2(1+x^2)} \end{array} \right]$$

(u-sub: $w = 1+x^2$)

$$= -\frac{\arctan x}{2(1+x^2)} \Big|_0^t + \frac{1}{2} \int_0^t \frac{dx}{(1+x^2)^2}$$

Let $x = \tan \theta$. $1+x^2 = \sec^2 \theta$.
 $dx = \sec^2 \theta d\theta$

$$= -\frac{\arctan t}{2(1+t^2)} + \frac{1}{2} \int_0^{\arctan t} \frac{\sec^2 \theta d\theta}{\sec^4 \theta} = -\frac{\arctan t}{2(1+t^2)} + \frac{1}{2} \int_0^{\arctan t} \cos^2 \theta d\theta$$

$$= \frac{-\arctan t}{2(1+t^2)} + \left[\frac{1}{4} \theta + \frac{1}{8} \sin 2\theta \right]_0^{\arctan t} = \frac{-\arctan t}{2(1+t^2)} + \frac{1}{4} \arctan t + \frac{1}{8} \sin(2 \tan^{-1}(t))$$

Taking the limit as $t \rightarrow \infty$, we find $\arctan t \rightarrow \frac{\pi}{2}$.

$$= 0 + \frac{\pi}{8} + 0 = \frac{\pi}{8}$$