Linear Algebra

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Usual way we talk about matrices

A real $m \times n$ matrix A is a grid of numbers A_{ij} with $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

Matrix multiplication is an operation that takes two matrices A, B of dimensions $l \times m$ and $m \times n$ and returns a third matrix C of dimensions $l \times n$ by the following rule:

$$C_{ij} = \sum_{k=1}^{m} A_{ik} B_{kj}$$

Matrices as linear maps

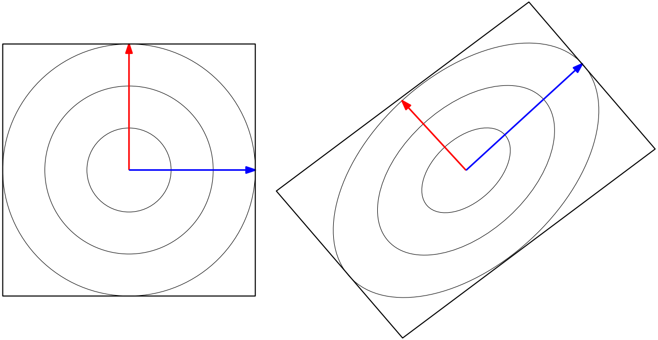
A real $m \times n$ matrix A is a function that eats vectors in $\mathbb{R}^n$ and spits out vectors in $\mathbb{R}^m$.

The entry A_{ij} is the number in the j-th coordinate of the output when your input is the i-th unit vector.

Matrix multiplication takes two of these functions A, B, for which B eats vectors in $\mathbb{R}^n$ and spits out vectors in $\mathbb{R}^m$, and A eats vectors in $\mathbb{R}^m$ and spits out vectors of $\mathbb{R}^l$, and it describes the function composition C which acts by sending the vector from $\mathbb{R}^n$ through B then A.

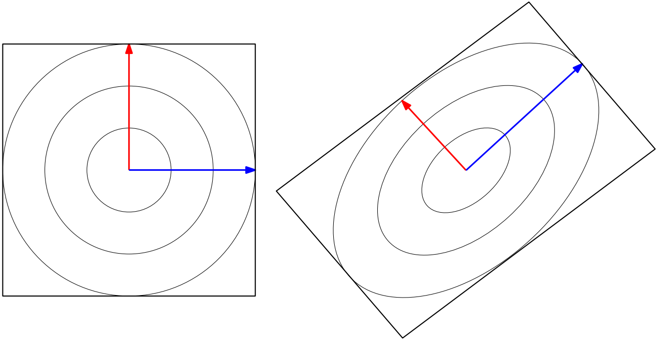
Matrices as linear maps - Picture

The following map from $\mathbb{R}^2$ to $\mathbb{R}^2$ rotates and scales the basis vectors.



The corresponding matrix $A = \begin{pmatrix} 1 & -1/2 \\ 1 & 1/2 \end{pmatrix}$.

Matrices as linear maps - Picture



The corresponding matrix $A = \begin{pmatrix} 1 & -1/2 \\ 1 & 1/2 \end{pmatrix}$.

The first column tells us the output for input $(1, 0)$, and the second column gives the output for input $(0, 1)$.

Linearity

A matrix defines a *linear map*. This means

$$A(c\vec{v} + d\vec{w}) = c \cdot A\vec{v} + d \cdot A\vec{w}.$$

In particular, scalar multiplication and addition of vectors can be done before applying the function A or after, and you'll get the same result!

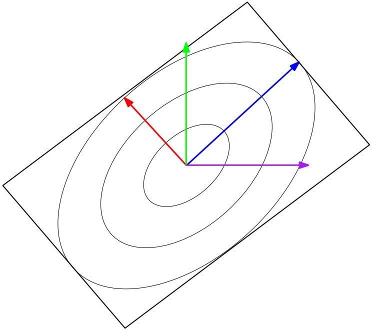
Identity Matrix

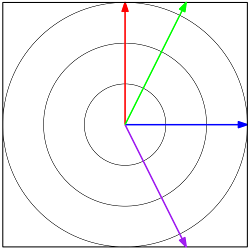
What matrix I takes vectors in $\mathbb{R}^n$ and spits out the same vector?

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & 1 & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Inverse Matrix

If a matrix A is a function eating vectors in $\mathbb{R}^n$ and spitting out vectors in $\mathbb{R}^n$, the inverse matrix A^{-1} eats the output and spits out the input. $A^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ -1 & 1 \end{pmatrix}$.





Inverse Sanity Check

If A is an $m \times n$ matrix instead of an $n \times n$ square matrix, does the inverse exist?

- If $m > n$, the image of the smaller space does not fill up the larger space. A matrix $\hat{A}$ which acts as an inverse on the image of A exists, but it is not unique.
- If $n > m$, the larger space gets squashed into the smaller space, so many vectors get mapped to the same vector.

Not all square matrices are invertible

A square matrix is invertible if and only if its *determinant* is nonzero. We will study determinants in the next lecture. An example of a non-invertible matrix is:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 1 & 1 \end{pmatrix}$$

In terms of the linear map, if the matrix is not invertible, the map puts vectors into a smaller-dimensional space. For example, the image of this matrix is 2-dimensional.

Linear algebraic equations in Matrix Form

The following is a linear system of equations, because every terrm only has a single x variable in degree 1:

$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & = & b_2 \\ a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & = & b_3 \end{array}$$

It can be summarized as $A\vec{x} = \vec{b}$.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Linear algebraic equations in Matrix Form

The system $A\vec{x} = \vec{b}$ can also be considered in terms of A as a function on vector spaces.

i.e. *What vector $\vec{x}$ gives $\vec{b}$ as the output when you run it through A?*

If A is an invertible square matrix, then you can just run $\vec{b}$ through A^{-1} to find the desired $\vec{x}$.

Using MATLAB to define matrices

The first matrix definition below explicitly lists entries. The others define special matrices. Try them on MATLAB.

```
A = [1,2;5,6];
B = zeros(3,4);
C = ones(2,3);
D = eye(5);
```

Using MATLAB to solve linear systems

Let A be an $n \times n$ square matrix defining a system of linear algebraic equations as $A\vec{x} = \vec{b}$.

The following commands in MATLAB solve for x.

```
x = A\b;
x = inv(A)*b;
```

Other matrix operations

The following are also matrix operations in MATLAB, for A, B both $m \times n$ matrices, and c a scalar:

1. `A + B`, coordinate-wise addition.
2. `A .* B`, coordinate-wise multiplication.
3. `c * A`, scalar multiplication by a matrix.
4. `A'`, transpose of the matrix.