MATH 1B Fall 2011
Quiz 11

Name: ____________________

Roots	General Solution
$r_1 \neq r_2$, real	$y(x) = c_1e^{r_1x} + c_2e^{r_2x}$
$r_1 = r_2 = r$, real	$y(x) = c_1e^{rx} + c_2xe^{rx}$
r_1, r_2 complex, $\alpha \pm i\beta$	$y(x) = e^{\alpha x}(c_1\cos(\beta x) + c_2\sin(\beta x))$

Consider the differential equation:

$$y'' - y = e^x$$

1. (*3 points*) Use **power series** to find a complementary solution for this differential equation.

y_c is the solution to $y_c'' - y_c = 0$.

If $y_c = \sum_{n=0}^{\infty} c_n x^n$, then $y_c'' = \sum_{n=0}^{\infty} (n+1)(n+2)c_{n+2}x^n$.

$$\Rightarrow y_c'' - y_c = \sum_{n=0}^{\infty}(n+1)(n+2)c_{n+2}x^n - \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty}\left[(n+1)(n+2)c_{n+2} - c_n\right]x^n = 0.$$

$$\Rightarrow (n+1)(n+2)c_{n+2} - c_n = 0 \quad \forall n \geq 0.$$

$$\Rightarrow c_{n+2} = \frac{c_n}{(n+1)(n+2)} \quad n \geq 0 \quad \Leftrightarrow \quad c_n = \frac{c_{n-2}}{n(n-1)} \text{ for } n \geq 2.$$

$\Rightarrow$ $c_0 = c_0$

$c_1 = c_1$

$c_2 = \frac{c_0}{1 \cdot 2}$

$c_3 = \frac{c_1}{1 \cdot 2 \cdot 3}$

$c_4 = \frac{c_2}{3 \cdot 4} = \frac{c_0}{1 \cdot 2 \cdot 3 \cdot 4}$

$c_5 = \frac{c_3}{4 \cdot 5} = \frac{c_1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$

$$c_n = \begin{cases} \frac{c_0}{n!} & n \text{ even} \\ \frac{c_1}{n!} & n \text{ odd} \end{cases} \Rightarrow \quad c_{2n} = \frac{c_0}{(2n)!} \quad c_{2n+1} = \frac{c_1}{(2n+1)!}$$

$$\Rightarrow \boxed{\begin{aligned} y &= c_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} + c_1 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \\ &= c_0 \cosh x + c_1 \sinh x \\ &= \left(\frac{c_0 + c_1}{2}\right)e^x + \left(\frac{c_0 - c_1}{2}\right)e^{-x}. \end{aligned}}$$

2. (*3 points*) Use **the method of undetermined coefficients** to find a particular solution to this differential equation.

$y_c = c_1 e^{x} + c_2 e^{-x}$. So, y_p cannot be Ae^{x}; instead,

we take Axe^{x}. $\Rightarrow y_p' = Axe^{x} + Ae^{x} \Rightarrow y_p'' = Axe^{x} + 2Ae^{x}$.

$y_p'' - y_p = Axe^{x} + 2Ae^{x} - Axe^{x} = 2Ae^{x} = e^{x} \Rightarrow A = \frac{1}{2}$.

So, $\boxed{y_p = \frac{1}{2}xe^{x}}$

3. (*3 points*) Use **the method of variation of parameters** to find the particular solution.

$y_c = c_1 e^{x} + c_2 e^{-x}$. So we take $y_p = u_1(x) e^{x} + u_2(x) e^{-x}$.

with (1) $u_1' e^{x} + u_2' e^{-x} = 0$.

(2) $u_1' e^{x} - u_2' e^{-x} = e^{x}$

(1) + (2) : $2u_1' e^{x} = e^{x} \Rightarrow u_1' = \frac{1}{2} \Rightarrow u_1 = \frac{1}{2}x$.

Subbing in $u_1' = \frac{1}{2}$, $\frac{1}{2}e^{x} + u_2' e^{-x} = 0$

$\Rightarrow u_2' e^{-x} = -\frac{1}{2} \cdot e^{x} \Rightarrow u_2' = -\frac{1}{2}e^{2x} \Rightarrow u_2 = -\frac{1}{4}e^{2x}$.

Then, $y_p = \left(\frac{1}{2}x\right)e^{x} + \left(-\frac{1}{4}e^{2x}\right)e^{-x} = \boxed{\frac{1}{2}xe^{x} - \frac{1}{4}e^{x}}$