Review Sheet - Anna & Ethan

Ch. 11: 1. $\sum_{n=1}^{\infty} (-1)^n \tan^{-1}\left(\frac{1}{n}\right)$ *

To test for convergence, use Alternating Series Test.

i) $\tan^{-1}\left(\frac{1}{n}\right) > 0$ since $\frac{1}{n} > 0$. $\Rightarrow (-1)^n b_n$ alternating.

ii) $\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2} > 0$. Since $\frac{1}{n+1} < \frac{1}{n} \Rightarrow$

$\tan^{-1}\left(\frac{1}{n+1}\right) < \tan^{-1}\left(\frac{1}{n}\right) \Rightarrow b_n$ decreasing.

iii) $\lim_{n\to\infty} \tan^{-1}\left(\frac{1}{n}\right) = \tan^{-1}(0) = 0$. ✓

So, $\sum (-1)^n \tan^{-1}\left(\frac{1}{n}\right)$ converges by AST.

For absolute convergence, note: $\tan^{-1}\left(\frac{1}{n}\right) = \frac{1}{n} - \frac{1}{n^3 \cdot 3} + \frac{1}{n^5 \cdot 5} \cdots$

$\Rightarrow \tan^{-1}\left(\frac{1}{n}\right) \approx \frac{1}{n}$ for large n.

So, we use LCT with $\frac{1}{n}$: $\lim_{n\to\infty} \frac{\tan^{-1}\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{1}{n} - \frac{1}{3n^3} + \frac{1}{5n^5} \cdots}{\frac{1}{n}}$

$= \lim_{n\to\infty} 1 - \frac{1}{3n^2} + \frac{1}{5n^4} = 1 > 0$. Since $\sum \frac{1}{n}$ diverges,

so does $\sum \tan^{-1}\left(\frac{1}{n}\right)$, so:

$\boxed{\sum (-1)^n \tan^{-1}\left(\frac{1}{n}\right) \text{ is conditionally convergent.}}$

Final Exam, 2010

6. c) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \sqrt{1-\cos\frac{1}{n}}$: AC, CC, or D?

We use Alternating Series Test:

i) Let $b_n = \sqrt{1-\cos\frac{1}{n}}$. $b_n > 0$ by definition. ✓

ii) $\dfrac{d}{dx}\left(\sqrt{1-\cos\frac{1}{x}}\right) = \left[\dfrac{1}{2\sqrt{1-\cos\frac{1}{x}}} \cdot \sin\left(\frac{1}{x}\right)\cdot\left(-\frac{1}{x^2}\right)\right] < 0$ for all

x greater than 1 $\Rightarrow$ b_n decreasing.

iii) $\lim\limits_{n\to\infty} \sqrt{1-\cos\frac{1}{n}} = \sqrt{1-\cos 0} = 0$. ✓

$\Rightarrow$ $\sum (-1)^n \sqrt{1-\cos\frac{1}{n}}$ converges by AST.

We note that $\cos\frac{1}{n} = 1 - \frac{1}{2n^2} + \frac{1}{4!\,n^4} - \cdots \Rightarrow \cos\frac{1}{n} \approx 1 - \frac{1}{2n^2}$

for large n. $\Rightarrow \sqrt{1-\cos\frac{1}{n}} \approx \sqrt{\frac{1}{2n^2}} = \frac{1}{n\sqrt{2}}$.

LCT with $\frac{1}{n}$ returns $\lim\limits_{n\to\infty} \dfrac{\sqrt{1-\cos\frac{1}{n}}}{\frac{1}{n}} = \frac{1}{\sqrt{2}} > 0$.

$\Rightarrow$ Since $\sum\frac{1}{n}$ diverges, so does $\sum\sqrt{1-\cos\frac{1}{n}}$.

so, $\boxed{\displaystyle\sum_{n=1}^{\infty} (-1)^n \sqrt{1-\cos\left(\tfrac{1}{n}\right)} \text{ is conditionally convergent.}}$

$\boxed{\text{CC}}$